SOME MICROLOCAL ASPECTS OF PERVERSE COHERENT SHEAVES AND EQUIVARIANT D-MODULES

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ABSTRACT

We discuss microlocal aspects of two types of sheaves which are of interest to geometric representation theory: perverse coherent sheaves and equivariant D-modules.

The category of (constructible) perverse sheaves on a complex variety is characterized by exactness of the microlocal stalks (or vanishing cycles) functor. We prove an analogue of this characterization for the category of perverse coherent sheaves on a scheme with a group action. The main idea is to understand microlocal stalks via local cohomology along half-dimensional ("Lagrangian") subvarieties. We define "measuring subvarieties" as an analogue of these subvarieties in the coherent setting and show how they can be used to characterize perverse coherent sheaves.

The second part of this thesis is dedicated to understanding the support theory (in the sense of [BIK]) of equivariant D-modules. We discuss how to compute the Hochschild cohomology of the category of D-modules on a quotient stack via a relative compactification of the diagonal morphism. We then apply this construction to the case of torus-equivariant D-modules and describe the Hochschild cohomology as the cohomology of a D-module on the loop space of the quotient stack.

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CONTENTS

1

INTRODUCTION

This thesis consists of to parts, concerning perverse coherent sheaves and equivariant D-modules respectively. Both parts are motivated by trying to understand microlocal properties of sheaves of importance in geometric representation theory. While the specific results of the two parts look unrelated on the first glance, they are linked by the notion of singular support for coherent sheaves. We will now give a brief overview of the ideas and results, discussing in particular how we understand "microlocal" in each case.

1.1. MICROLOCAL THEORY OF PERVERSE COHERENT SHEAVES

The first part of this thesis is motivated by two theories that came to recent prominence in algebraic geometry and in particular geometric representation theory:

- The theory of perverse coherent sheaves, independently introduced by Bezrukavnikov (following Deligne) [AB], Kashiwara [K1], Gabber [G] and Bridgeland [B5].

- A theory of "singular support" for coherent sheaves, worked out by Arinkin and Gaitsgory for their formulation of the Geometric Langlands Conjecture [AG], based on earlier work by Benson, Iyengar and Krause [BIK].

Removing the word "coherent", these two theories are by now classical and indispensable tools in the study of constructible sheaves. They can also be beautifully combined to elucidate the microlocal nature of perverse sheaves [KS]. Thus the question naturally arises whether the coherent versions of these theories can be combined too.

Coming form representation theory, we will be primarily concerned with the definition of perverse coherent sheaves by Bezrukavnikov. We will recall the basic constructions and theorems of this theory below. However we should note from the outset that the Arinkin–Gaitsgory theory of singular support does not interact well with these perverse coherent sheaves. For example the perverse t-structure on SL_2-equivariant coherent sheaves on $\mathbb{A}^2$ is non-trivial, but the singular support of coherent sheaves on either $\mathbb{A}^2$ or the stack $\mathbb{A}^2/\,\mathrm{SL}_2$ is always trivial. Thus we have to understand the microlocal nature of perverse coherent sheaves in a different way.

Perverse (constructible) sheaves

Before discussing our results, let us briefly review the theory of perverse sheaves in the setting that is closest to the perverse coherent sheaves of [AB]. For readability, we will restrict to the complex case and the middle perversity.

Thus we let X be a complex variety and fix a stratification $\mathfrak{S}$ of X by smooth complex subvarieties. Attached to this setup we have the bounded derived category $\mathbf{D}^b_{\mathfrak{S}}(X)$ of $\mathfrak{S}$-constructible sheaves, i.e. the full subcategory of the category of constructible sheaves on X consisting of sheaves which are local systems along each stratum. The *perverse t-structure* on this category is then given by the two full subcategories

$$\begin{aligned} {}^p\mathbf{D}^{\leq 0}_{\mathfrak{S}}(X) &= \{\mathscr{F} \in \mathbf{D}^b_{\mathfrak{S}}(X) : i_S^*\mathscr{F} \in \mathbf{D}^{\leq -\frac{1}{2}\dim_{\mathbb{R}} S}(S) \text{ for all } S \in \mathfrak{S}\}, \\ {}^p\mathbf{D}^{\geq 0}_{\mathfrak{S}}(X) &= \{\mathscr{F} \in \mathbf{D}^b_{\mathfrak{S}}(X) : i_S^!\mathscr{F} \in \mathbf{D}^{\geq -\frac{1}{2}\dim_{\mathbb{R}} S}(S) \text{ for all } S \in \mathfrak{S}\}, \end{aligned} \tag{1.1}$$

where $i_S \colon S \hookrightarrow X$ is the inclusion. That this is indeed a t-structure was proved by Beilinson, Bernstein and Deligne in [BBD]. In particular this means that the category

$$\mathrm{Perv}_{\mathfrak{S}}(X) = {}^p\mathbf{D}^{\leq 0}_{\mathfrak{S}}(X) \cap {}^p\mathbf{D}^{\geq 0}_{\mathfrak{S}}(X)$$

is Abelian. It is called the category of *(middle) perverse sheaves* on X with respect to $\mathfrak{S}$. The category $\mathrm{Perv}_{\mathfrak{S}}(X)$ is intimately connected to both the singularities of X (via intersection cohomology) and, if X is smooth, the category of D-modules on X (via the Riemann-Hilbert correspondence). It has many nice properties, of which we want to mention the following two which are of particular importance in the coherent analogue.

- The Verdier duality functor on $\mathbf{D}^b_{\mathfrak{S}}(X)$ is compatible with the perverse t-structure and in particular restricts to an involution of $\mathrm{Perv}_{\mathfrak{S}}(X)$.
- The simple objects of $\mathrm{Perv}_{\mathfrak{S}}(X)$ are in bijection with pairs $(S, \mathscr{L})$ consisting of a stratum $S \in \mathfrak{S}$ and an irreducible local system $\mathscr{L}$ on S.

Perverse coherent sheaves

The definition of perverse sheaves can be translated to coherent sheaves with some modification. We will again only discuss the middle perversity here. The general definition will be reviewed in Section 2.2.

We let X be a scheme of finite type over a field k. We would like to pick a stratification of X and consider the category of coherent sheaves whose restriction to each stratum is a vector bundle. Unfortunately this is not a triangulated subcategory of $\mathbf{D}^b_{\mathrm{coh}}(X)$. For example for a generic function f the cone of $\mathscr{O}_X \xrightarrow{\cdot f} \mathscr{O}_X$ will not be a vector bundle on the open stratum.

The solution to this problem is to introduce a group action and replace "smoothness along a stratification" by "equivariantness". Thus we let G be an affine algebraic group over k acting on X and consider the bounded derived category $\mathbf{D}^b_{\mathrm{coh}}(X)^G = \mathbf{D}^b_{\mathrm{coh}}(X/G)$ of G-equivariant coherent sheaves on X. Instead of the set of strata $\mathfrak{S}$ we will look at the set $X^{G,\mathrm{gen}}$ of generic points of G-equivariant subschemes of X.

In the constructible setting we required the stratification to be complex, so that we could take half of the (real) dimension. Now we will require directly that all G-orbits are even-dimensional. The main example of this situation is the nilpotent cone of a semi-simple algebraic group with the adjoint action.

The only piece remaining to translate from the constructible setting are the restriction functors. For this it turns out that the $\mathscr{O}$-module functors do not give the correct definition and we have to use the k-module functors instead. Thus, if $\imath_x : \{x\} \to X$ is the inclusion of $x \in X^{G,\mathrm{gen}}$ and $\mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X)^G$, we let $\boldsymbol{\imath}_x^* \mathscr{F} = \mathscr{F}_x$ be the (derived) functor of talking stalks. Similarly, we let $\boldsymbol{\imath}_x^! \mathscr{F} = \boldsymbol{\imath}_x^* \Gamma_{\overline{\{x\}}} \mathscr{F}$ be the derived functor of local cohomology.

All together, in analogy with (1.1) we now define the two full subcategories

$$
\begin{aligned}
{}^p\mathbf{D}^{\leq 0}(X)^G &= \{\mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X) : \boldsymbol{\imath}_x^* \mathscr{F} \in \mathbf{D}^{\leq -\frac{1}{2}\dim x}(\mathscr{O}_x) \text{ for all } x \in X^{G,\mathrm{gen}}\}, \\
{}^p\mathbf{D}^{\geq 0}(X)^G &= \{\mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X) : \boldsymbol{\imath}_x^! \mathscr{F} \in \mathbf{D}^{\geq -\frac{1}{2}\dim x}(\mathscr{O}_x) \text{ for all } x \in X^{G,\mathrm{gen}}\},
\end{aligned}
$$

of $\mathbf{D}^b_{\mathrm{coh}}(X)^G$. This is indeed a t-structure [AB]. The heart of this t-structure is called the category of *(middle) perverse sheaves* on X. It is again an Abelian category with lots of interesting features. In particular we have the same properties as above:

- Grothendieck-Serre duality is an involution of the category of perverse coherent sheaves.
- The simple objects are indexed by pairs $(O, \mathscr{V})$, where O is a G-orbit and $\mathscr{V}$ is an irreducible G-equivariant vector bundle on O (or equivalently an irreducible representation of the stabilizer of G on O).

Bezrukavnikov used the second property to establish a bijection between pairs (O, V) of a nilpotent orbit O and an irreducible representation V of the stabilizer of G on O, and the set Λ^+ of dominant weight of G, thus proving a conjecture of Lusztig [B3].

Microlocal theory (constructible version)

The definition of perverse sheaves seems very mysterious. Why would the category defined this way be important and have nice features? One answer to this question was proposed by Kashiwara and Schapira [KS] in their microlocal viewpoint.

A good way to understand the microlocal nature of perverse sheaves is via the vanishing cycles functor. For this let f be a (local) holomorphic function on X. Then the vanishing cycles $\varphi_f : \mathbf{D}^b_{\mathrm{constr}}(X) \to \mathbf{D}^b_{\mathrm{constr}}(f^{-1}(0))$ restrict to a functor on the corresponding categories of perverse sheaves

$$\varphi_f : \mathrm{Perv}(X) \to \mathrm{Perv}(f^{-1}(0)).$$

If in addition we let f be a stratified Morse function with critical point x, then taking corresponding *microlocal stalks*

$$\left(\varphi_f(-)\right)_x : \mathbf{D}^b_{\mathfrak{S}}(X) \to \mathbf{D}(\mathbb{C})$$

sends perverse sheaves to vector spaces concentrated in degree 0. In fact, this property characterizes perverse sheaves among all $\mathfrak{S}$-constructible sheaves on X [J].

In a further reformulation, we can take local cohomology along the unstable manifold Z of $\mathfrak{Ref}$ and require that this is concentrated in degree 0. We note that

$$\dim_{\mathbb{R}} Z = \frac{1}{2} \dim_{\mathbb{R}} X.$$

Microlocal theory (coherent version)

This last observation can be translated to the world of coherent sheaves. For this purpose we will define *measuring subvarieties* of X. Roughly speaking, these are subvarieties of X which intersects each G-orbit in a half dimensional subvariety. We will give the precise definition in Chapter 3, where we will also prove the following analogue of the above characterization of perverse (constructible) sheaves. See Theorem 3.8 for the exact statement and some variants.

Theorem 1.1. *A sheaf $\mathcal{F} \in \mathbf{D}^b_{\mathrm{coh}}(X)^G$ is perverse if and only if $\Gamma_Z \mathcal{F}$ is concentrated in cohomological degree 0 for sufficiently many measuring subvarieties Z of X.*

1.2. HOCHSCHILD COHOMOLOGY OF CATEGORIES OF D-MODULES

Given a manifold X and a category of sheaves on X, microlocal geometry asks whether these sheaves can be localized not just on X but also with respect to codirections, i.e. on the cotangent space T^*X. For example, for constructible sheaves this leads to the notion of microsupport discussed in detail in [KS]. More generally, given a category of sheaves on a space X, we can ask whether it is possible to localize them on some space that is strictly larger than X itself.

Even more generally one can ask the following question: Given a k-linear category $\mathbf{C}$, can one find a space over which $\mathbf{C}$ localizes? For co-complete compactly generated triangulated categories one answer is provided by [BIK]: To each map from a graded-commutative ring R to the center of $\mathbf{C}$ the authors associate the *triangulated support* functor supp_R, assigning to each object $A \in \mathbf{C}$ a subset $\mathrm{supp}_R A \subseteq \operatorname{Spec} R$. This construction can be used to unify various theories of support in different areas of mathematics (though it does not yield the microlocal support of constructible sheaves).

We are led to consider the universal algebra acting on the category with this construction, i.e. the *Hochschild cohomology* of $\mathbf{C}$. For a complete (pre-triangulated) dg category $\mathbf{C}$ the Hochschild cohomology is the dg algebra of derived endomorphisms of the identity functor of $\mathbf{C}$:

$$\mathrm{HH}^\bullet(\mathbf{C}) = \mathbf{R}\mathrm{Hom}(\mathrm{Id}_{\mathbf{C}}, \mathrm{Id}_{\mathbf{C}}) = \mathrm{Hom}_{\mathbf{Funct}(\mathbf{C},\mathbf{C})}(\mathrm{Id}_{\mathbf{C}}, \mathrm{Id}_{\mathbf{C}}).$$

The ring $R = \bigoplus \mathrm{HH}^{2n}(\mathrm{HH}^\bullet(\mathbf{C})$ is commutative and hence one can define for each $A \in \mathbf{C}$ the support $\mathrm{supp}_R A$ as a subset of $\operatorname{Spec} R$. Thus understanding the Hochschild cohomology of a dg category can be an important step to understanding the category itself.

This construction, applied to the category of (ind-)coherent sheaves on a (quasismooth, dg-) scheme yields the singular support of coherent sheaves which already served as the motivation for the first part of this thesis. More concretely, Arinkin and Gaitsgory used singular support for the category $\mathbf{IndCoh}(\mathrm{LocSys}_G)$ in their formulation of the geometric Langlands conjecture [AG]. By Langlands duality, one should then have a matching support theory for the category $\mathbf{DMod}(\mathrm{Bun}_G)$ and the question arises whether it is possible to formulate this theory in a way that is intrinsic to D-modules.

A first step to this – and also a problem of independent interest – is to understand the Hochschild cohomology of the category $\mathbf{DMod}(\mathbf{X})$ of D-modules on a stack $\mathbf{X}$. We will review the general setup and basic properties of D-modules on (QCA) stacks in Chapter 4. The upshot is that we have an isomorphism of dg algebras

$$\mathrm{HH}^\bullet(\mathbf{DMod}(\mathbf{X})) \cong \mathrm{Hom}_{\mathbf{DMod}(X\times X)}(\Delta_*\omega_{\mathbf{X}}, \Delta_*\omega_{\mathbf{X}}), \tag{1.2}$$

where $\Delta\colon \mathbf{X}\times\mathbf{X}\to\mathbf{X}$ is the diagonal morphism and $\omega_{\mathbf{X}}$ is the dualizing module. In particular if $\mathbf{X}$ is a (separated) scheme, then Δ is a closed embedding and (Δ^*, Δ_*) adjunction combined with Kashiwara's Lemma show that $\mathrm{HH}^\bullet(\mathbf{DMod}(\mathbf{X}))$ is isomorphic to the de Rham cohomology of $\mathbf{X}$. However, if $\mathbf{X}$ is not an algebraic space (and hence Δ is not proper) then the situation becomes more complicated.

By Verdier duality and adjunction we can always rewrite (1.2) as

$$\mathrm{HH}^\bullet(\mathbf{DMod}(\mathbf{X})) \cong \mathrm{Hom}_{\mathbf{DMod}(X)}(k_{\mathbf{X}}, \Delta^!\Delta_!k_{\mathbf{X}})^{\mathrm{op}} = \Gamma_{\mathrm{dR}}(\mathbf{X}, \Delta^!\Delta_!k_{\mathbf{X}})^{\mathrm{op}}.$$

It is now tempting to look at the Cartesian square

$$\begin{tikzcd} \mathscr{L}\mathbf{X} \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathbf{X} \arrow[d, "\Delta"] \\ \mathbf{X} \arrow[r, "\Delta"] & \mathbf{X}\times\mathbf{X} \end{tikzcd}$$

where

$$\mathscr{L}\mathbf{X} = \mathbf{X} \underset{\mathbf{X}\times\mathbf{X}}{\times} \mathbf{X}$$

is the (derived) loop space of $\mathbf{X}$ and try to express the Hochschild cohomology as the cohomology of some sheaf on $\mathscr{L}\mathbf{X}$. Naively we could expect the existence of an isomorphism

$$\Gamma_{\mathrm{dR}}(\mathbf{X}, \Delta^!\Delta_!k_{\mathbf{X}}) \cong \Gamma_{\mathrm{dR}}(\mathbf{X}, p_{2,!}p_1^!k_{\mathbf{X}}). \tag{1.3}$$

Unfortunately, the two sides are in general not isomorphic (the stack $\mathbf{X} = \mathbb{P}^1/\mathbb{A}^1$ is an easy counter-example).

In Chapter 5 we will investigate how to quantify the cone of the morphism

$$p_{2,!}p_1^!k_{\mathbf{X}} \to \Delta^!\Delta_!k_{\mathbf{X}}$$

and thus the failure of the naive isomorphism (1.3) to hold. As an application, we will prove the following theorem, giving a class of stacks where (1.3) is indeed an isomorphism.

Theorem 1.2. *Let $G \cong \mathbb{G}_m^n$ be a torus acting locally linearly on a scheme X of finite type over k. Then there is a canonical isomorphism of algebras*

$$\mathrm{HH}^\bullet(\mathbf{DMod}(X/G)) \cong \Gamma_{\mathrm{dR}}(X/G,\, p_{2,!}p_1^! k_{X/G})^{\mathrm{op}},$$

where the algebra structure on $\Gamma_{\mathrm{dR}}(X/G,\, p_{2,!}p_1^! k_{X/G})$ is induced by the groupoid structure on $\mathscr{L}(X/G)$.

I

A microlocal description of perverse coherent sheaves

2

PREREQUISITES

Throughout this part we will be concerned with a scheme X with an action by an affine group scheme G. We assume that X and G are both of finite type over a field k.

We will make use of the usual notations for derived categories. Thus $\mathbf{D}_{\mathrm{qc}}(X)$ is the derived category of the Abelian category of quasi-coherent sheaves on X and $\mathbf{D}_{\mathrm{coh}}(X)$ is its full subcategory of complexes with coherent cohomology. More generally $\mathbf{D}(X)$ is the derived category of sheaves of k-modules on X. For a ring R, the derived category of R-modules will be denoted by $\mathbf{D}(R)$. To avoid cluttering the notation we will also usually suppress the signifiers $\mathbb{R}$ and $\mathbb{L}$ on functors between derived categories.

If Y is a subscheme of X we will always write $\imath_Y$ for the inclusion $Y \hookrightarrow X$.

2.1. OPERATIONS ON COHERENT SHEAVES

We will be mainly concerned with the category $\mathbf{D}^b_{\mathrm{coh}}(X)^G$, the bounded derived category of G-equivariant coherent sheaves on X. For the reader familiar with stacks, this is the same category as the bounded derived category of coherent sheaves on the quotient stack $[X/G]$. It is also equivalent to the full subcategory of $\mathbf{D}^b_{\mathrm{qc}}(X)^G$ consisting of complexes with coherent cohomology [AB, Corollary 2.11]. There is a forgetful functor

$$\mathrm{Forget}\colon \mathbf{D}^b_{\mathrm{coh}}(X)^G \to \mathbf{D}^b_{\mathrm{coh}}(X)$$

to the non-equivariant bounded derived category of coherent sheaves on X. We will frequently apply functors defined on the latter category to equivariant sheaves without explicitly mentioning the intervening forgetful functor.

Let Z be a closed subscheme of X. Then there are functors $\imath_Z^!$ and $\imath_Z^*$ from $\mathbf{D}^b_{\mathrm{coh}}(X)$ to $\mathbf{D}^b_{\mathrm{coh}}(Z)$, defined by

$$\imath_Z^*(-) = \mathcal{O}_Z \otimes_{\mathcal{O}_X} - \qquad \text{and} \qquad \imath_Z^!(-) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z, -).$$

We note again that the symbols $\otimes$ and $\mathcal{H}om$ denote the corresponding derived functors. If U is an open subscheme of X then the functor $\imath_U^* = \imath_U^!$ is the restriction functor $\mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(U)$. For a general locally closed subscheme, the restriction functors are defined be composing the above functors.

We will also need the corresponding k-module functors, which we will denote by bold letters. In particular, if Z is a closed subspace of X (as a topological space), then we set

$$\boldsymbol{\imath}_Z^!(-) = \mathcal{H}om_{k_X}(k_Z, -).$$

Following [H, Variation 3 in IV.1] we write

$$\Gamma_Z = \boldsymbol{\imath}_{Z,*}\boldsymbol{\imath}_Z^!.$$

Again we note that these functors should be seen as functors between the derived categories. It is well known that if $\mathcal{F}$ is a (complex of) quasi-coherent sheaf(s) on X, then $\Gamma_Z\mathcal{F}$ is again quasi-coherent [SGA2, Corollaire II.3].

Let x be a (not necessarily closed) point of X. We will write $\boldsymbol{\imath}_x^*$ for the functor of talking stalks at x and set $\boldsymbol{\imath}_x^! = \boldsymbol{\imath}_x^*\Gamma_{\overline{\{x\}}}$. As noted before, we will apply all of these functors to equivariant sheaves without explicitly mentioning the forgetful functor.

Finally we will need the Grothendieck-Verdier duality functor on $\mathbf{D}^b_{\mathrm{coh}}(X)^G$. It is defined exactly as in the non-equivariant situation (cf. [H, Chapter V]). Thus, an equivariant dualizing complex on X is an object $\mathcal{DC} \in \mathbf{D}^b_{\mathrm{coh}}(X)^G$ such each object $\mathcal{F} \in \mathbf{D}^b_{\mathrm{coh}}(X)^G$ is $\mathcal{DC}$-reflexive, i.e. such that the natural transformation

$$\mathcal{F} \to \mathcal{H}om(\mathcal{H}om(\mathcal{F}, \mathcal{DC}), \mathcal{DC}) \qquad (\mathcal{F} \in \mathbf{D}^b_{\mathrm{coh}}(X))$$

is an isomorphism. We write $\mathbb{D}$ for the endofunctor $\mathcal{H}om(-, \mathcal{DC})$ of $\mathbf{D}^b_{\mathrm{coh}}(X)^G$. Since $\mathcal{H}om$ commutes with the forgetful functor, if $\mathcal{DC}$ is an equivariant dualizing complex, then $\mathrm{Forget}(\mathcal{DC}) \in \mathbf{D}^b_{\mathrm{coh}}(X)$ is a (non-equivariant) dualizing complex. Under our assumptions on X there always exists an equivariant dualizing complex $\mathcal{DC}$ [AB, Theorem 2.18]. Using [H, V.7], we will further assume that for each (not necessarily closed) point $x \in X$ the complex $\boldsymbol{\imath}_x^!\mathcal{DC}$ is concentrated in cohomological degree $-\dim x$.

We finish this section with two lemmas about vanishing of local cohomology above or below a certain degree.

Lemma 2.1. *Let $\mathcal{F}$ be a coherent sheaf on X and let x be a closed point of X. Then $\boldsymbol{\imath}_x^*\mathbb{D}\mathcal{F} \in \mathbf{D}^{\geq 0}(\mathcal{O}_x)$ if and only if $\boldsymbol{\imath}_x^!\mathcal{F} \in \mathbf{D}^{\leq 0}(\mathcal{O}_x)$.*

Proof. The proof of this lemma is essentially the same as the one of [AB, Lemma 3.3(a)]. Concretely, by [H, V.6], there is an isomorphism of functors

$$\boldsymbol{\imath}_z^!(-) \cong \mathrm{Hom}_{\mathcal{O}_x}(\mathbb{D}(-), \mathcal{I}_x),$$

where $\mathcal{I}_x$ is the injective hull of the residue field of $\mathcal{O}_x$. The statement now follows from the fact that $\mathrm{Hom}_{\mathcal{O}_x}(-, \mathcal{I}_x)$ is exact and kills no finitely generated $\mathcal{O}_x$-module [H, V.5]. □

Lemma 2.2. *Let $\mathcal{F}$ be a coherent sheaf on X and let Z be a closed subvariety of X. Then $\Gamma_Z\mathcal{F} \in \mathbf{D}^{\geq 0}(Z)$ if and only if $\imath_Z^!\mathcal{F} \in \mathbf{D}^{\geq 0}(Z)$.*

Proof. This is the equivalence of (i) and (ii) in [SGA2, Proposition VII.1.2] with $Y = S = Z$, $G = \mathcal{F}$, $F = \mathcal{O}_Z$ and $n = 1$. □

2.2. PERVERSE COHERENT SHEAVES

We keep the general assumptions on X and the dualizing complex $\mathscr{DC}$. We write $X^{G,\mathrm{gen}}$ for the set of generic points of G-stable subschemes of X. It is a subset of the topological space of X and we will consider it with the induced topology.

By a *perversity* we mean a function $p\colon \{0,\dots,\dim X\} \to \mathbb{Z}$. For $x \in X^{G,\mathrm{gen}}$ we abuse notation and set $p(x) = p(\dim x)$. Then $p\colon X^{G,\mathrm{gen}} \to \mathbb{Z}$ is a perversity function in the sense of [B2]. Note that we insist that $p(x)$ only depends on the dimension of x.

A perversity is called *monotone* if it is decreasing and *comonotone* if the *dual perversity* $\overline{p}(n) = -n - p(n)$ is decreasing. It is *strictly monotone* (resp. *strictly comonotone*) if for all $x, y \in X^{G,\mathrm{gen}}$ with $\dim x < \dim y$ one has $p(x) > p(y)$ (resp. $\overline{p}(x) > \overline{p}(y)$). Note that a strictly monotone perversity is not necessarily strictly decreasing (e.g. if X only has even-dimensional G-orbits).

Following [AB] we now have all ingredients to define the perverse t-structure on $\mathbf{D}^b_{\mathrm{coh}}(X)^G$.

Definition 2.3. Given a perversity p we define the following full subcategories of $\mathbf{D}^b_{\mathrm{coh}}(X)^G$:

$$
\begin{aligned}
{}^p\mathbf{D}^{\leq 0}(X)^G &= \{\mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X) : \imath_x^*\mathscr{F} \in \mathbf{D}^{\leq p(x)}(\mathscr{O}_x) \text{ for all } x \in X^{G,\mathrm{gen}}\},\\
{}^p\mathbf{D}^{\geq 0}(X)^G &= \{\mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X) : \imath_x^!\mathscr{F} \in \mathbf{D}^{\geq p(x)}(\mathscr{O}_x) \text{ for all } x \in X^{G,\mathrm{gen}}\}.
\end{aligned}
$$

Theorem 2.4 ([AB, Theorem 3.10])**.** *If p is monotone and comonotone, then $({}^p\mathbf{D}^{\leq 0}(X)^G, {}^p\mathbf{D}^{\leq 0}(X)^G)$ defines a t-structure on $\mathbf{D}^b_{\mathrm{coh}}(X)^G$.*

This t-structure is called the *perverse t-structure* with respect to p on $\mathbf{D}^b_{\mathrm{coh}}(X)^G$. Objects in its heart are called *perverse coherent sheaves* (with respect to p on X).

The perverse t-structure is compatible with duality, exchanging the perversity p with its dual.

Lemma 2.5 ([AB, Lemma 3.3])**.** *Let p be any perversity. Then*

$$\mathbb{D}({}^p\mathbf{D}^{\leq 0}(X)^G) = {}^{\overline{p}}\mathbf{D}^{\geq 0}(X)^G.$$

Example 2.6. The best-studied case of perverse coherent sheaves is the nilpotent cone N of a semi-simple algebraic group G with the adjoint action. It is well known that there are finitely many G-orbits on N, all of which are even dimensional. Thus there is a *middle perversity* given by

$$p(x) = \overline{p}(x) = -\frac{1}{2}\dim x, \qquad x \in N^{G,\mathrm{gen}}.$$

This t-structure has important applications in geometric representation theory, for example [B3] and [BM]. For an overview of the theory of perverse coherent sheaves on nilpotent cones and the related category of exotic sheaves we refer to [A]. ○

For later use we state the following variant of the Grothendieck Finiteness Theorem [SGA2, Théorème 2.1]. The given formulation is from [AB, Corollary 3.12], where the reader can find a short proof using the theory of perverse coherent sheaves.

Theorem 2.7. *Let p be a monotone and comonotone perversity on X. Let $x \in X^{G,\mathrm{gen}}$, set $U = X - \bar{x}$ and let $j\colon U \hookrightarrow X$ be the inclusion. Let $\mathscr{F} \in {}^p\mathbf{D}^{\geq 0}(U)^G$. Then $H^n(j_*\mathscr{F})$ is coherent for $n \leq p(x) - 2$.*

3

MEASURING SUBVARIETIES

Assumption 3.1. We will always assume that p is a monotone and comonotone perversity function. Then Theorem 2.4 guarantees the existence the perverse t-structure on $\mathbf{D}^b_{\mathrm{coh}}(X)^G$.

3.1. SOME REFORMULATIONS

In this section we will give some reformulations of the perverse t-structure from Definition 2.3. The equivalent conditions are inspired by Kashiwara's definition of a (non-equivariant) perverse t-structure on $\mathbf{D}^b_{\mathrm{coh}}(X)^G$ in [K1].

Proposition 3.2. *Let $\mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X)^G$. The following are equivalent:*

(i) $\mathscr{F} \in {}^p\mathbf{D}^{\leq 0}(X)^G$*, i.e.* $\imath_x^*\mathscr{F} \in \mathbf{D}^{\leq p(x)}(\mathscr{O}_x)$ *for all* $x \in X^{G,\mathrm{gen}}$*;*

(ii) $p(\dim \operatorname{supp} H^k(\mathscr{F})) \geq k$ *for all* k*.*

A crucial fact that we will implicitly use quite often in the following arguments is that the support of a coherent sheaf is always closed. In particular, this means that if x is a generic point and $\mathscr{F}$ a coherent sheaf, then $\imath_x^*\mathscr{F} = 0$ if and only if $\mathscr{F}|_U = 0$ for some open set U intersecting $\overline{x}$.

Proof. First let $\mathscr{F} \in {}^p\mathbf{D}^{\leq 0}(X)^G$ and assume for contradiction that there exists an integer k such that $p(\dim \operatorname{supp} H^k(\mathscr{F})) < k$. Let x be the generic point of an irreducible component of maximal dimension of $\operatorname{supp} H^k(\mathscr{F})$. Then $H^k(\imath_x^*\mathscr{F}) \neq 0$. But on the other hand, $\imath_x^*\mathscr{F} \in \mathbf{D}^{\leq p(x)}(\mathscr{O}_x)$ and $p(x) = p(\dim \operatorname{supp} H^k(\mathscr{F})) < k$, yielding a contradiction.

Conversely assume that $p(\dim \operatorname{supp} H^k(\mathscr{F})) \geq k$ for all k and let $x \in X^{G,\mathrm{gen}}$. If $H^k(\imath_x^*\mathscr{F}) \neq 0$, then $\dim x \leq \dim \operatorname{supp} H^k(\mathscr{F})$. Thus monotonicity of the perversity implies that $\mathscr{F} \in {}^p\mathbf{D}^{\leq 0}(X)^G$. $\square$

Proposition 3.3. *Let $\mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X)^G$ and let p be strictly monotone.*

(i) $\mathscr{F} \in {}^p\mathbf{D}^{\geq 0}(X)^G$*, i.e.* $\imath_x^!\mathscr{F} \in \mathbf{D}^{\geq p(x)}(\mathscr{O}_x)$ *for all* $x \in X^{G,\mathrm{gen}}$*;*

(ii) $\Gamma_{\overline{x}}\mathscr{F} \in \mathbf{D}^{\geq p(x)}(X)$ *for all* $x \in X^{G,\mathrm{gen}}$*;*

(iii) $\Gamma_Y\mathscr{F} \in \mathbf{D}^{\geq p(\dim Y)}(X)$ *for all G-invariant closed subvarieties Y of X;*

(iv) $\dim\left(\bar{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}\mathscr{F})\right)\right) \leq -p(x) - k$ *for all* $x \in X^{G,\mathrm{gen}}$ *and all* k.

Proof. The implications from (iii) to (ii) and (ii) to (i) are trivial and the equivalence of (ii) and (iv) follows from Lemma 3.4 below. Thus we only need to show that (i) implies (iii). So assume that $\mathscr{F} \in {}^p\mathbf{D}^{\geq 0}(X)^G$. We induct on the dimension of Y.

If $\dim Y = 0$, then $\Gamma(X, \Gamma_Y \mathscr{F}) = \bigoplus_{y \in Y^{G,\mathrm{gen}}} \imath_y^! \mathscr{F}$ and thus $\Gamma_Y \mathscr{F} \in \mathbf{D}^{\geq p(0)}(X)$ by assumption.

Now let $\dim Y > 0$. We first assume that Y is irreducible with generic point $x \in X^{G,\mathrm{gen}}$. Let k be the smallest integer such that $H^k(\Gamma_{\bar{x}}\mathscr{F}) \neq 0$ and assume that $k < p(x)$. We will show that this implies that $H^k(\Gamma_{\bar{x}}\mathscr{F}) = 0$, giving a contradiction.

We first show that $H^k(\Gamma_{\bar{x}}\mathscr{F})$ is coherent. Let $j\colon X - \bar{x} \hookrightarrow X$ and consider the distinguished triangle

$$\Gamma_{\bar{x}}\mathscr{F} \to \mathscr{F} \to j_* j^* \mathscr{F} \xrightarrow{+1}.$$

Applying cohomology to it we get an exact sequence

$$H^{k-1}(j_* j^* \mathscr{F}) \to H^k(\Gamma_{\bar{x}}\mathscr{F}) \to H^k(\mathscr{F}).$$

By assumption, $k-1 \leq p(x)-2$, so that $H^{k-1}(j_* j^* \mathscr{F})$ is coherent by the Grothendieck Finiteness Theorem 2.7. As $H^k(\mathscr{F})$ is coherent by definition, this implies that $H^k(\Gamma_{\bar{x}}\mathscr{F})$ also has to be coherent.

Set $Z = \operatorname{supp} H^k(\Gamma_{\bar{x}}\mathscr{F})$. Then, since $\imath_x^* H^k(\Gamma_{\bar{x}}\mathscr{F}) = H^k(\imath_x^! \mathscr{F})$ vanishes, Z is a proper closed subset of $\bar{x}$. We consider the distinguished triangle

$$H^k(\Gamma_{\bar{x}}\mathscr{F})[-k] \to \Gamma_{\bar{x}}\mathscr{F} \to \tau_{>k}\Gamma_{\bar{x}}\mathscr{F} \xrightarrow{+1},$$

and apply Γ_Z to it:

$$\Gamma_Z H^k(\Gamma_{\bar{x}}\mathscr{F})[-k] = H^k(\Gamma_{\bar{x}}\mathscr{F})[-k] \to \Gamma_Z \mathscr{F} \to \Gamma_Z \tau_{>k}\Gamma_{\bar{x}}\mathscr{F} \xrightarrow{+1}.$$

Since $\dim Z < \dim x$, we can use the induction hypothesis and monotonicity of p to deduce that $\Gamma_Z \mathscr{F}$ is in degrees at least $p(\dim Z) \geq p(x) > k$. Clearly $\Gamma_Z \tau_{>k}\Gamma_{\bar{x}}\mathscr{F}$ is also in degrees larger than k. Hence $H^k(\Gamma_{\bar{x}}\mathscr{F})$ has to vanish.

If Y is not irreducible, let Y_1 be an irreducible component of Y and Y_2 be the union of the other components. Then there is a Mayer-Vietoris distinguished triangle

$$\Gamma_{Y_1 \cap Y_2}\mathscr{F} \to \Gamma_{Y_1}\mathscr{F} \oplus \Gamma_{Y_2}\mathscr{F} \to \Gamma_Y \mathscr{F} \xrightarrow{+1},$$

where $\Gamma_{Y_1 \cap Y_2}\mathscr{F} \in \mathbf{D}^{\geq p(\dim Y_1 \cap Y_2)}(X) \subseteq \mathbf{D}^{\geq p(\dim Y)+1}(X)$ (by the induction hypothesis and strict monotonicity of p) and $\Gamma_{Y_1}\mathscr{F}$ and $\Gamma_{Y_2}\mathscr{F}$ are in $\mathbf{D}^{\geq p(\dim Y)}(X)$ by induction on the number of components of Y. Thus $\Gamma_Y \mathscr{F} \in \mathbf{D}^{\geq p(\dim Y)}(X)$ as required. $\square$

Lemma 3.4. *Let* $\mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X)$*,* Z *a closed subset of* X*, and* n *an integer. Then* $\Gamma_Z \mathscr{F} \in \mathbf{D}^{\geq n}_{\mathrm{qc}}(X)$ *if and only if* $\dim(Z \cap \operatorname{supp}(H^k(\mathbb{D}\mathscr{F}))) \leq -k - n$ *for all* k.

This lemma extends [K1, Proposition 5.2] to singular varieties. The proof is essentially the same as for the smooth case, but we will include it here for completeness.

Proof. By [SGA2, Proposition VII.1.2], $\Gamma_Z \mathscr{F} \in \mathbf{D}^{\geq n}_{\mathrm{qc}}(X)$ if and only if

$$\mathscr{H}\!om(\mathscr{G}, \mathscr{F}) \in \mathbf{D}^{\geq n}_{\mathrm{qc}}(()X) \tag{3.1}$$

for all $\mathscr{G} \in \mathbf{Coh}(X)$ with $\operatorname{supp} \mathscr{G} \subseteq Z$. Let $d(n) = -n$ be the dual standard perversity. Then by [B2, Lemma 5a], (3.1) holds if and only if $\mathbb{D}\mathscr{H}\!om(\mathscr{G}, \mathscr{F}) \in {}^{d}\mathbf{D}^{\leq -n}(X)^G$. By [H, Proposition V.2.6], $\mathbb{D}\mathscr{H}\!om(\mathscr{G}, \mathscr{F}) = \mathscr{G} \otimes_{\mathscr{O}_X} \mathbb{D}\mathscr{F}$, so that by Proposition 3.2 we need to show that

$$\dim \operatorname{supp} H^k\left(\mathscr{G} \otimes_{\mathscr{O}_X} \mathbb{D}\mathscr{F}\right) \leq -k - n$$

for all k. By [K1, Lemma 5.3] (whose proof does not use the smoothness assumption) this is equivalent to

$$\dim\left(Z \cap \operatorname{supp} H^k(\mathbb{D}\mathscr{F})\right) \leq -k - n$$

for all k, completing the proof. □

3.2. PERVERSE COHERENT SHEAVES VIA MEASURING SUBVARIETIES

Assumption 3.5. From now on we will assume that the G-action on X has finitely many orbits.

Definition 3.6. Let p be a perversity. A *p-measuring subvariety* of X is a closed subvariety Z of X such that

$$\dim(\overline{x} \cap Z) = \dim x + p(x)$$

for each $x \in X^{G,\mathrm{gen}}$ with $\overline{x} \cap Z \neq \emptyset$. If in addition $\overline{x} \cap Z$ is a set-theoretic local complete intersection in $\overline{x}$ for each $x \in X^{G,\mathrm{gen}}$, then Z is called a *strong* p-measuring subvariety. A *(strong) p-measuring collection of subvarieties* of X is a collection $\mathfrak{M}$ of (strong) p-measuring subvarieties Z such that for each $x \in X^{G,\mathrm{gen}}$ there exists $Z \in \mathfrak{M}$ with $\overline{x} \cap Z \neq \emptyset$.

Remark 3.7. Let Z be a p-measuring subvariety. The condition on p-measuring subvarieties can be rewritten as $\dim(\overline{x} \cap Z) = -\overline{p}(x)$ and $\operatorname{codim}_{\overline{x}}(\overline{x} \cap Z) = -p(x)$. Thus comonotonicity of p ensures that if $\dim y \leq \dim x$ then $\dim(\overline{y} \cap Z) \leq \dim(\overline{x} \cap Z)$. Monotonicity of p then further says that $\operatorname{codim}_{\overline{y}}(\overline{y} \cap Z) \leq \operatorname{codim}_{\overline{x}}(\overline{x} \cap Z)$.

We clearly have $0 \leq \dim(\overline{x} \cap Z) \leq \dim x$ and hence $-\dim x \leq p(x) \leq 0$. We will show in Theorem 3.10 that the condition $-\dim x \leq p(x) \leq 0$ is sufficient for the (local) existence of a strong p-measuring collection.

Theorem 3.8. *Let $\mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X)^G$.*

(i) *Assume p is strictly monotone and that X has a p-measuring collection $\mathfrak{M}$. Then the following are equivalent.*

  (a) $\mathscr{F} \in {}^{p}\mathbf{D}^{\geq 0}(X)^G$;

(b) $i_Z^! \mathscr{F} \in \mathbf{D}^{\geq 0}(Z)$ *for all* $Z \in \mathfrak{M}$*;*

(c) $\Gamma_Z \mathscr{F} \in \mathbf{D}^{\geq 0}(X)$ *for all* $Z \in \mathfrak{M}$*.*

(ii) *Assume that* p *is strictly comonotone and that* X *has a* $\overline{p}$*-measuring collection* $\mathfrak{M}$*. Then the following are equivalent.*

(a) $\mathscr{F} \in {}^p\mathbf{D}^{\leq 0}(X)^G$*;*

(b) $\Gamma_z i_Z^* \mathscr{F} \in \mathbf{D}^{\leq 0}(Z)$ *for all* $Z \in \mathfrak{M}$ *and all* $z \in Z$*.*

(iii) *Assume that* X *has a strong* p*-measuring collection* $\mathfrak{M}$*. Then the following are equivalent.*

(a) $\mathscr{F} \in {}^p\mathbf{D}^{\leq 0}(X)^G$*;*

(b) $\Gamma_Z \mathscr{F} \in \mathbf{D}^{\leq 0}(X)$ *for all* $Z \in \mathfrak{M}$*.*

In particular, if p *is strictly monotone and* X *has a strong* p*-measuring collection* $\mathfrak{M}$*, then* $\mathscr{F}$ *is perverse with respect to* p *if and only if* $\Gamma_Z \mathscr{F}$ *is cohomologically concentrated in degree* 0 *for each* $Z \in \mathfrak{M}$*.*

Proof of Theorem 3.8(i). The equivalence of (c) and (b) follows directly from Lemma 2.2. We will prove the equivalence of (a) and (c).

By Proposition 3.3, $\mathscr{F} \in {}^p\mathbf{D}^{\geq 0}(X)^G$ if and only if

$$\dim\left(\overline{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) \leq -p(x) - k \qquad \text{for all } x \in X^{G,\mathrm{gen}} \text{ and all } k. \tag{3.2}$$

Using Lemma 3.4 for $\Gamma_Z \mathscr{F} \in \mathbf{D}^{\geq 0}(X)$, we see that we have to show the equivalence of (3.2) with

$$\dim\left(Z \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) \leq -k \qquad \text{for all } k \text{ and all } Z \in \mathfrak{M}.$$

Since there are only finitely many orbits, this is in turn equivalent to

$$\dim\left(Z \cap \overline{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) \leq -k \qquad \forall\, x \in X^{G,\mathrm{gen}},\, k \text{ and } Z \in \mathfrak{M}. \tag{3.3}$$

We will show the equivalence for each fixed k separately. Let us first show the implication from (3.2) to (3.3). Since $H^k(\mathbb{D}\mathscr{F})$ is G-equivariant and there are only finitely many G-orbits, it suffices to show (3.3) assuming that $\dim x \leq \dim \operatorname{supp} H^k(\mathbb{D}F)$ and $\overline{x} \cap \operatorname{supp} H^k(\mathbb{D}F) \neq \emptyset$. Then $\dim\left(\overline{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) = \dim \overline{x}$. Thus,

$$\begin{aligned}\dim\left(Z \cap \overline{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) &\leq \dim(Z \cap \overline{x}) = p(x) + \dim x = \\ &p(x) + \dim\left(\overline{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) \leq p(x) - p(x) - k = -k.\end{aligned}$$

Conversely, assume that (3.3) holds for k. If $\overline{x} \cap \operatorname{supp} H^k(\mathbb{D}F) = \emptyset$, then (3.2) is trivially true. Otherwise choose a p-measuring Z that intersects $\operatorname{supp} H^k(\mathbb{D}F)$. First assume that $\overline{x}$ is contained in $\operatorname{supp} H^k(\mathbb{D}F)$. Then

$$\dim\left(\bar{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) = \dim x = -p(x) + \dim(Z \cap \bar{x}) = \\ -p(x) + \dim\left(Z \cap \bar{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) \leq -p(x) - k.$$

Otherwise $\bar{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right) = \bar{y}$ for some $y \in X^{G,\mathrm{gen}}$ with $\dim y < \dim x$. Then (3.2) holds for y in place of x and hence

$$\dim\left(\bar{x} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) = \dim\left(\bar{y} \cap \operatorname{supp}\left(H^k(\mathbb{D}F)\right)\right) \leq -p(y) - k \leq -p(x) - k$$

by monotonicity of p. □

We now obtain the second part of the theorem by a duality argument from the first one.

Proof of Theorem 3.8(ii). Let $\mathscr{F} \in {}^p\mathbf{D}^{\leq 0}(X)^G$. By Lemma 2.5 this is equivalent to $\mathbb{D}\mathscr{F} \in {}^{\bar{p}}\mathbf{D}^{\geq 0}(X)^G$. By part (i) this is in turn equivalent to $\imath_z^* \imath_Z^! \mathbb{D}\mathscr{F} \in \mathbf{D}^{\geq 0}(\mathscr{O}_z)$ for all $Z \in \mathfrak{M}$ and $z \in Z$. The sheaf $\imath_Z^! \mathbb{D}\mathscr{F} = \mathbb{D}\imath_Z^* \mathscr{F}$ is coherent, so that the statement now follows from Lemma 2.1. □

The following lemma encapsulates the central argument of the proof of the remaining part of Theorem 3.8.

Lemma 3.9. *Let $\mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X)^{G,\heartsuit}$ be a G-equivariant coherent sheaf on X (i.e. a complex concentrated in degree 0), let p be a monotone perversity and let n be an integer. Assume that X has enough p-measuring subvarieties and let $\mathfrak{M}$ be a p-measuring family of subvarieties of X. Then the following are equivalent:*

(i) $p(\dim \operatorname{supp} \mathscr{F}) \geq n$;

(ii) $H^i(\Gamma_Z \mathscr{F}) = 0$ *for all* $i \geq -n+1$ *and all* $Z \in \mathfrak{M}$.

Proof. Since $\operatorname{supp} \mathscr{F}$ is always a union of the closure of orbits, we can restrict to the support and assume that $\operatorname{supp} \mathscr{F} = X$.

First assume that $p(\dim X) = p(\dim \operatorname{supp} \mathscr{F}) \geq n$. Using a Mayer-Vietoris argument it suffices to check condition (ii) in the case that X is irreducible. By the definition of a p-measuring subvariety and monotonicity of p, this implies that, up to radical, Z can be locally defined by at most $-n$ equations. Thus $H^i(\Gamma_Z \mathscr{F}) = 0$ for $i > -n$ [BS, Theorem 3.3.1].

Now assume conversely that $H^i(\Gamma_Z \mathscr{F}) = 0$ for all $i \geq -n+1$ and all measuring subvarieties $Z \in \mathfrak{M}$. We have to show that $p(\dim X) \geq n$. Set $d = \dim X$. Choose any p-measuring subvariety $Z \in \mathfrak{M}$ that intersects a maximal component of X non-trivially. Then $\operatorname{codim}_X Z = -p(d)$. We will show that $H^{-p(d)}(\Gamma_Z \mathscr{F}) \neq 0$ and hence $p(d) \geq n$ by assumption. Take some affine open subset U of X such that $U \cap Z$ is non-empty, irreducible and of codimension $-p(d)$ in U. It suffices to show that the cohomology is non-zero in U. Thus we can assume without loss of generality that X is affine, say $X = \operatorname{Spec} A$, and Z is irreducible. Write $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of A. By flat base change [BS, Theorem 4.3.2],

$$\Gamma(X, H^{-p(d)}(\Gamma_Z \mathscr{F}))_{\mathfrak{p}} = \left(H_{\mathfrak{p}}^{-p(d)}(\Gamma(X, \mathscr{F}))\right)_{\mathfrak{p}} = H_{\mathfrak{p}_{\mathfrak{p}}}^{-p(d)}(\Gamma(X, \mathscr{F})_{\mathfrak{p}})$$

Since $\dim \operatorname{supp} \mathscr{F} = \dim X = d$, the dimension of the $A_{\mathfrak{p}}$-module $\Gamma(X, \mathscr{F})_{\mathfrak{p}}$ is $-p(d)$. Thus by the Grothendieck non-vanishing theorem [BS, Theorem 6.1.4] $H^{-p(d)}_{\mathfrak{p}_{\mathfrak{p}}}(\Gamma(X, \mathscr{F})_{\mathfrak{p}}) \neq 0$ and hence $\Gamma(X, H^{-p(d)}(\Gamma_Z \mathscr{F})) \neq 0$ as required. □

Proof of Theorem 3.8(iii). We use the description of ${}^p\mathbf{D}^{\leq 0}(X)^G$ given by Proposition 3.2, i.e.

$${}^p\mathbf{D}^{\leq 0}(X)^G = \left\{ \mathscr{F} \in \mathbf{D}^b_{\mathrm{coh}}(X)^G : p\left(\dim\left(\operatorname{supp} H^n(\mathscr{F})\right)\right) \geq n \text{ for all } n \right\}.$$

We induct on the largest k such that $H^k(\mathscr{F}) \neq 0$ to show that $\mathscr{F} \in {}^p\mathbf{D}^{\leq 0}(X)^G$ if and only if $\Gamma_Z \mathscr{F} \in \mathbf{D}^{\leq 0}(X)$ for all p-measuring subvarieties $Z \in \mathfrak{M}$.

The equivalence is trivial for $k \ll 0$. For the induction step note that there is a distinguished triangle

$$\tau_{<k} \mathscr{F} \to \mathscr{F} \to H^k(\mathscr{F})[-k] \xrightarrow{+1}.$$

Applying the functor Γ_Z and taking cohomology we obtain an exact sequence

$$\begin{aligned} \cdots \to H^1(\Gamma_Z(\tau_{<k}\mathscr{F})) \to H^1(\Gamma_Z \mathscr{F}) \to H^{k+1}(\Gamma_Z(H^k(\mathscr{F}))) \to \\ H^2(\Gamma_Z(\tau_{<k}\mathscr{F})) \to H^2(\Gamma_Z \mathscr{F}) \to H^{k+2}(\Gamma_Z(H^k(\mathscr{F}))) \to \cdots. \end{aligned}$$

By induction, $H^j(\Gamma_Z(\tau_{<k}\mathscr{F}))$ vanishes for $j \geq 1$ so that $H^j(\Gamma_Z \mathscr{F}) \cong H^{k+j}(\Gamma_Z(H^k(\mathscr{F})))$ for $j \geq 1$. Thus the statement follows from Lemma 3.9. □

3.3. Existence of strong measuring subvarieties

Of course, for Theorem 3.8 to have any content, one needs to show that X has enough p-measuring subvarieties. The next theorem shows that at least for affine varieties there are always enough measuring subvarieties whenever p satisfies the obvious conditions (see Remark 3.7).

Theorem 3.10. *Assume that X is affine and the perversity p is monotone and comonotone and satisfies $-n \leq p(n) \leq 0$ for $n \in \{0, \ldots, \dim X\}$. Then X has enough strong p-measuring subvarieties.*

Proof. Let $X = \operatorname{Spec} A$. We induct on the dimension d. More precisely, we induct on the following statement:

> There exists a closed subvariety Z_d of X such that for all $x \in X^{G,\mathrm{gen}}$ the following holds:
>
> - $Z_d \cap \overline{x} \neq \emptyset$ and $Z_d \cap \overline{x}$ is a set-theoretic local complete intersection in $\overline{x}$;
> - if $\dim x \leq d$, then $\dim(\overline{x} \cap Z_d) = p(x) + \dim x$;
> - if $\dim x > d$, then $\dim(\overline{x} \cap Z_d) = p(d) + \dim x$.

We set $p(-1) = 0$. The statement is trivially true for $d = -1$, e.g. take $Z = X$. Assume that the statement is true for some $d \geq -1$. We want to show it for $d + 1 \leq \dim X$.

If $p(d) = p(d+1)$, then $Z_{d+1} = Z_d$ works. Otherwise, by (co)monotonicity, $p(d+1) = p(d)-1$. Set $S = \bigcup\{\overline{x} \in X^{G,\mathrm{gen}} : \dim x \leq d\}$. Since there are only finitely many orbits, we can choose a function f such that f vanishes identically on S, $V(f)$ does not share a component with Z_d and $V(f)$ intersects every $\overline{x}$ with $\dim x > d$. Then $Z_{d+1} = Z_d \cap V(f)$ satisfies the conditions. $\square$

II

Hochschild cohomology of D-modules on torus quotient stacks

4

PREREQUISITES

We fix an algebraically closed base field k of characteristic 0. All stacks in this thesis are assumed to be algebraic QCA stacks over k. As we will summarize in Section 4.1, the QCA condition ensures that the category of D-modules on stacks is well-behaved. In particular for any stack $\mathbf{X}$ we have:

- The diagonal morphism $\Delta\colon \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ is schematic.
- There exists a scheme Z with a smooth and surjective map $Z \to \mathbf{X}$.
- $\mathbf{X}$ is quasi-compact.
- The automorphism groups of the geometric points of $\mathbf{X}$ are affine.
- The loop space (or inertia stack) $\mathscr{L}\mathbf{X} = \mathbf{X} \times_{\mathbf{X}\times\mathbf{X}} \mathbf{X}$ is of finite presentation over $\mathbf{X}$.

The first two conditions ensure that the stack is algebraic, the other three that it is quasi-compact with affine automorphism group (QCA). For details on QCA stacks we refer to [DG1]. Every quotient of a scheme of finite type over k by an affine algebraic group is a QCA stack, and we will be mainly interested in these.

In order to correctly define categories of D-modules on stacks it is necessary to work with dg-categories. We refer to [K2] for an introduction to dg categories. It is often convenient to regard (pretriangulated) dg categories as k-linear stable $(\infty, 1)$-categories [L1; L2], which can be done via the nerve construction [C; F]. We will switch between those two languages without explicitly mentioning the intervening constructions and apply results from [L2] to dg categories. Fortunately, a superficial knowledge of dg/∞-categories should be sufficient for reading this thesis.

4.1. D-MODULES ON STACKS

We will be primarily concerned with D-modules on (quotient) stacks. Unfortunately there is currently no comprehensive text available that covers all the basic constructions and properties of D-modules on stacks (or even the *dg category* of D-modules on schemes). Thus we collect all the relevant properties (without proof) in this section. The upshot is that the familiar "six functors formalism" essentially works for holonomic D-modules and schematic morphisms of stacks.

The category of D-modules on a stack $\mathbf{X}$ can be either constructed via descent [BD; DG1] or equivalently as ind-coherent sheaves on the de Rham space of $\mathbf{X}$ [GR2]. While the first construction is more "hands on", the second construction is often more useful from a theoretical point of view. It is explained in detail in the upcoming book [GR1] (see also [FG] for an overview). Many basic properties of the category $\mathbf{DMod}(\mathbf{X})$ are explored in [DG1] and most of the following assertions are taken from there.

For any morphism $f: \mathbf{X} \to \mathbf{Y}$ the constructions yield a continuous functor $f^!: \mathbf{DMod}(\mathbf{Y}) \to \mathbf{DMod}(\mathbf{X})$ and (after some work) a not necessarily continuous functor $f_*: \mathbf{DMod}(\mathbf{X}) \to \mathbf{DMod}(\mathbf{Y})$. If $p: \mathbf{X} \to \mathrm{pt}$ is the structure map then we set

$$\Gamma_{\mathrm{dR}}(\mathbf{X}, -) = p_*(-): \mathbf{DMod}(\mathbf{X}) \to \mathbf{Vect}.$$

The functor Γ_{dR} is representable by a D-module $k_{\mathbf{X}}$, i.e.

$$\Gamma_{\mathrm{dR}}(\mathbf{X}, -) = \mathrm{Hom}_{\mathbf{DMod}(\mathbf{X})}(k_{\mathbf{X}}, -).$$

Again we note that $\Gamma_{\mathrm{dR}}(\mathbf{X}, -)$ is usually not continuous and hence the object $k_{\mathbf{X}}$ not compact.

Let $\Delta: \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ be the diagonal. The category $\mathbf{DMod}(\mathbf{X})$ has a monoidal structure given by the tensor product

$$\mathscr{F} \otimes \mathscr{G} = \Delta^!(\mathscr{F} \boxtimes \mathscr{G}).$$

The unit for this monoidal structure is $\omega_{\mathbf{X}} = p^! k$.

We will be mainly concerned with the subcategory of holonomic D-modules since they enjoy extended functoriality.

Definition 4.1. A D-module $\mathscr{F} \in \mathbf{DMod}(\mathbf{X})$ is called *holonomic* if $f^!\mathscr{F}$ is holonomic for any smooth morphism $f: Z \to \mathbf{X}$ from a scheme Z. The full subcategory of holonomic D-modules will be denoted $\mathbf{DMod}_{\mathrm{hol}}(\mathbf{X})$.

The following assertions mostly follow from their corresponding counterparts for schemes. We refer to [B4] for proofs in the case of non-smooth schemes.

Proposition 4.2. *Let $f: \mathbf{X} \to \mathbf{Y}$ be a schematic morphism. Then $f^!$ and f_* restrict to functors*

$$f^!: \mathbf{DMod}_{\mathrm{hol}}(\mathbf{Y}) \to \mathbf{DMod}_{\mathrm{hol}}(\mathbf{X}) \quad \textit{and} \quad f_*: \mathbf{DMod}_{\mathrm{hol}}(\mathbf{X}) \to \mathbf{DMod}_{\mathrm{hol}}(\mathbf{Y}).$$

The Verdier duality functor on schemes induces an involutive anti auto-equivalence

$$\mathbb{D}_{\mathbf{X}}: \mathbf{DMod}_{\mathrm{hol}}(\mathbf{X})^{\mathrm{op}} \to \mathbf{DMod}_{\mathrm{hol}}(\mathbf{X})$$

such that for each smooth morphism $Z \to \mathbf{X}$ of relative dimension d from a scheme Z one has

$$f^! \circ \mathbb{D}_{\mathbf{X}} \cong \mathbb{D}_Z \circ f^![-2d].$$

The Verdier duality functor then allows us to define the *non-standard functors* $f_!$ and f^* for any schematic morphism $f : \mathbf{X} \to \mathbf{Y}$ by

$$f^* = \mathbb{D}_{\mathbf{X}} \circ f^! \circ \mathbb{D}_{\mathbf{Y}} : \mathbf{DMod}_{\mathrm{hol}}(\mathbf{Y}) \to \mathbf{DMod}_{\mathrm{hol}}(\mathbf{X})$$

and

$$f_! = \mathbb{D}_{\mathbf{Y}} \circ f_* \circ \mathbb{D}_{\mathbf{X}} : \mathbf{DMod}_{\mathrm{hol}}(\mathbf{X}) \to \mathbf{DMod}_{\mathrm{hol}}(\mathbf{Y}).$$

We obtain adjoint pairs $(f_!, f^!)$ and (f^*, f_*). In some situations we can identify the non-standard functors with their standard counterparts. If f is smooth of relative dimension d then $f^* = f^![-2d]$. If f is proper then $f_! = f_*$ and in particular f_* is left adjoint to $f^!$. The objects $\omega_{\mathbf{X}}$ and $k_{\mathbf{X}}$ are always holonomic and

$$\mathbb{D}_{\mathbf{X}} \omega_{\mathbf{X}} = k_{\mathbf{X}}.$$

We have $k_{\mathbf{X}} = f^* k_{\mathbf{Y}}$ and if $\mathbf{X}$ is smooth, then $k_{\mathbf{X}} = \omega_{\mathbf{X}}[-2 \dim \mathbf{X}]$.

We will make use of the following lemma which follows from [DG1, Lemma 5.1.6].

Lemma 4.3. *For a smooth and schematic morphism f the functor $f^!$ is conservative.*

Proposition 4.4 ([GR1, III.4.2.1.3])**.** *Consider a Cartesian square*

$$\begin{tikzcd} \mathbf{Z} \arrow[r, "q"] \arrow[d, "p"] & \mathbf{X}_1 \arrow[d, "f"] \\ \mathbf{X}_2 \arrow[r, "g"] & \mathbf{Y} \end{tikzcd}$$

with schematic morphism f (and hence p). Then there is a base change isomorphism

$$p_* q^! \xrightarrow{\sim} g^! f_*$$

of functors from $\mathbf{DMod}(\mathbf{X_1})$ to $\mathbf{DMod}(\mathbf{X_2})$. If furthermore f (and hence p) is proper, then this isomorphism coincides with the natural transformation

$$p_* q^! \to p_* q^! f^! f_* = p_* p^! g^! f_* \to g^! f_*$$

induced by $(f_*, f^!)$ and $(p_*, p^!)$ adjunctions.

Proposition 4.5. *If $f : \mathbf{X} \to \mathbf{Y}$ is a schematic morphism then the projection formula holds, i.e. there is a functorial isomorphism*

$$\mathscr{F} \otimes f_*(\mathscr{G}) \cong f_*(f^! \mathscr{F} \otimes \mathscr{G})$$

for $\mathscr{F} \in \mathbf{DMod}(\mathbf{Y})$ and $\mathscr{G} \in \mathbf{DMod}(\mathbf{X})$.

Remark 4.6. Propositions 4.4 and 4.5 hold more generally when f is merely a “safe” morphism. Alternatively they hold in full generality after replacing f_* by the “renormalized de Rham pushforward”. We fill not use either notion in this thesis and refer the interested reader to [DG1].

For D-modules on stacks we have the usual recollement package. Let $i\colon \mathbf{Z} \hookrightarrow \mathbf{X}$ be a closed embedding and $j\colon \mathbf{U} \hookrightarrow \mathbf{X}$ the complementary open. We have adjoint pairs $(i_*, i^!)$ and $(j^!, j_*)$.

Proposition 4.7 ([GR2, Section 2.5])**.** *There is an exact triangle of functors*

$$i_* i^! \to \mathrm{Id} \to j_* j^!$$

on $\mathbf{DMod}(\mathbf{X})$*, the adjunction morphisms*

$$\mathrm{Id} \to i^! i_* \quad \textit{and} \quad j^! j_* \to \mathrm{Id}$$

are isomorphisms, the functors $j^! i_*$ *and* $i^! j_*$ *vanish and* i_* *and* j_* *are full embeddings.*

On holonomic D-modules we have the additional adjoint pairs (i^*, i_*) and $(j_!, j^*)$. By applying duality to Proposition 4.7 we obtain the distinguished triangle

$$j_! j^* \to \mathrm{Id} \to i_* i^*$$

and the identity $i^* j_! = 0$ on holonomic D-modules. Further, the functor $j_!$ is a full embedding $\mathbf{DMod}_{\mathrm{hol}}(\mathbf{U}) \hookrightarrow \mathbf{DMod}_{\mathrm{hol}}(\mathbf{X})$.

It is often useful to consider the pullback of a D-module on $\mathbf{X}$ to a smooth cover.

Definition 4.8. Let X be a scheme with an action of an algebraic group G and let $p\colon X \to X/G$ be the quotient map. The *monodromic* subcategory $\mathbf{DMod}(X)^{G-\mathrm{mon}} \subseteq \mathbf{DMod}(X)$ is the full subcategory generated by the essential image of $p^!\colon \mathbf{DMod}(X/G) \to \mathbf{DMod}(X)$ (or equivalently by the essential image of p^*).

Theorem 4.9 (Contraction principle [DG2, Proposition 3.2.2])**.** *Let X be a scheme with an action by $\mathbb{G}_m$ that extends to an action of the monoid $\mathbb{A}^1$. Let $i\colon X^0 \hookrightarrow X$ be the closed subscheme of $\mathbb{G}_m$-fixed points and let $\pi : X \to X^0$ be the contraction morphism induced by the $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1 \to \{0\}$. Then there is an isomorphism of functors*

$$i^* \cong \pi_*\colon \mathbf{DMod}(X)^{\mathbb{G}_m-\mathrm{mon}} \to \mathbf{DMod}(X^0).$$

4.2. Monads

We will deduce Theorem 1.2 from a isomorphism of monads on $\mathbf{DMod}(\mathbf{X})$. In this section we give a short introduction to the theory of monads and the specific constructions that we will use. However, in the interest of readability we will mainly do so informally, skipping over the intricacies of ∞-categories. The interested reader can find the correct ∞-categorical formulations in the given references.

Thus we think of a *monad* on a category $\mathbf{C}$ as consisting of a triple (T, η, μ), where $T\colon \mathbf{C} \to \mathbf{C}$ is an endofunctor of $\mathbf{C}$, and $\eta\colon \mathrm{Id}_{\mathbf{C}} \to T$ and $\mu\colon T \circ T \to T$ are natural transformations such that the diagrams

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \arrow[r, "T\eta"] \arrow[d, "\eta T"'] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \tag{4.1}$$

commute. Alternatively, we can think of T being a monoid in the category of endofunctors of $\mathbf{C}$ with the monoidal structure given by composition of endofunctors. This definition also gives the correct generalization to ∞-categories [L2, Definition 4.7.0.1].

Let X be an object of the category $\mathbf{C}$. Then T gives the vector space $\mathrm{Hom}_{\mathbf{C}}(X, TX)$ the structure of a dg algebra with multiplication map

$$(f, g) \mapsto \mu_X \circ Tf \circ g,$$

$$X \xrightarrow{g} TX \xrightarrow{Tf} T^2X \xrightarrow{\mu_X} TX.$$

The identities 4.1 ensure that the algebra is associative and unital.

The most common source of monads is from a pair of adjoint functors $F\colon \mathbf{C} \rightleftarrows \mathbf{D} :G$. One simply sets $T = G \circ F$ and η and μ are given by the adjunction morphisms

$$\mathrm{Id}_{\mathbf{C}} \to G \circ F = T \qquad \text{and} \qquad T^2 = G \circ (F \circ G) \circ F \to G \circ F = T.$$

We note that the correct construction in more complicated in the ∞-categorical case and refer to [L2, Section 4.7]. For any $X \in \mathbf{C}$ the algebra construction above gives an isomorphism of algebras

$$\mathrm{Hom}_{\mathbf{C}}(X, (GF)(X)) \cong \mathrm{Hom}_{\mathbf{D}}(FX, FX).$$

Another common way to obtain monads in geometry is via a groupoid. Recall that a groupoid $G_\bullet$ in stacks consists of a stack $\mathbf{G}_0$ of "objects" and a stack $\mathbf{G}_1$ of "morphisms" together with

- *source* and *target* maps $s, t\colon \mathbf{G}_1 \rightrightarrows \mathbf{G}_0$,
- a *unit* $e\colon G_0 \to G_1$,
- a *multiplication* (or *composition*) map $m\colon \mathbf{G}_1 \underset{s,\mathbf{G}_0,t}{\times} \mathbf{G}_1 \to \mathbf{G}_1$,
- an *inverse* map $\imath\colon \mathbf{G}_1 \to \mathbf{G}_1$,

such that

- $s \circ e = t \circ e = \mathrm{Id}_{\mathbf{G}_0}$,
- $s \circ m = s \circ p_2$ and $t \circ m = t \circ p_1$ (where $p_i\colon \mathbf{G}_1 \times_{s,\mathbf{G}_0,t} \mathbf{G}_1$ are the projection maps).

- m is associative,
- $\imath$ interchanges s and t and is an inverse for m,

where all identities have to be understood in the correct ∞-categorical way [L1, Section 6.1.2].

Example 4.10. For our purpose the most important example is the following: Let $f : \mathbf{X} \to \mathbf{S}$ be a morphism of stacks. We set $\mathbf{G}_0 = \mathbf{X}$ and $\mathbf{G}_1 = \mathbf{X} \times_{\mathbf{S}} \mathbf{X}$. The source and target maps are given by p_1 and p_2, the unit by the diagonal $\Delta : \mathbf{X} \to \mathbf{X} \times_{\mathbf{S}} \mathbf{X}$, the inverse by interchanging the factors and multiplication is $p_{13} : \mathbf{X} \times_{\mathbf{S}} \mathbf{X} \times_{\mathbf{S}} \mathbf{X} \to \mathbf{X} \times_{\mathbf{S}} \mathbf{X}$. ○

Let us for the moment assume that s (and hence t) is proper and schematic. In this case the maps e and m are also proper, since $s \circ e = \mathrm{Id}_{\mathbf{G}_0}$ and $s \circ m = s \circ p_2$ are proper. In particular the functors $e^! : \mathbf{DMod}(\mathbf{G}_1) \to \mathbf{DMod}(\mathbf{G}_0)$ and $m^! : \mathbf{DMod}(\mathbf{G}_1) \to \mathbf{DMod}(\mathbf{G}_1 \times_{\mathbf{G}_0} \mathbf{G}_1)$ have left adjoints given by e_* and m_* respectively. This allows us to give the endofunctor $T = s_* t^!$ of $\mathbf{DMod}(\mathbf{G}_0)$ the structure of a monad in the following way:

- By $(e_*, e^!)$-adjunction we have a transformation
$$\mathrm{Id} = (s \circ e)_* (t \circ e)^! = s_* e_* e^! t^! \to s_* t^! = T.$$
- Consider the following commutative diagram
$$\begin{tikzcd}
 & & \mathbf{G}_1 \arrow[dddll, bend right, "s"'] \arrow[dddrr, bend left, "t"] & & \\
 & & \mathbf{G}_1 \times_{\mathbf{G}_0} \mathbf{G}_1 \arrow[u, "m"] \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & & \\
 & \mathbf{G}_1 \arrow[dl, "s"'] \arrow[dr, "t"] & & \mathbf{G}_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\
\mathbf{G}_0 & & \mathbf{G}_0 & & \mathbf{G}_0
\end{tikzcd}$$
with Cartesian middle square. Proper base change and $(m_*, m^!)$-adjunction gives a transformation
$$T^2 = s_* t^! s_* t^! = (s \circ p_2)_* (t \circ p_1)^! = (s \circ m)_* (t \circ m)^! = s_* m_* m^! t^! \to s_* t^! = T.$$

In the non-∞-categorical setting one could easily check by hand that this is indeed a monad. To obtain the corresponding derived statement one applies an argument similar to [GR1, Section II.1.7.2]. We will discuss a version of this below.

Let now $f : \mathbf{X} \to \mathbf{Y}$ be schematic and proper. The Cartesian diagram

$$\begin{tikzcd} \mathbf{X} \times_{\mathbf{Y}} \mathbf{X} \arrow[r, "p_s"] \arrow[d, "p_t"] & \mathbf{X} \arrow[d, "f"] \\ \mathbf{X} \arrow[r, "f"] & \mathbf{Y} \end{tikzcd}$$

induces a groupoid with $\mathbf{G}_0 = \mathbf{X}$ and $\mathbf{G}_1 = \mathbf{X} \times_{\mathbf{Y}} \mathbf{X}$. The above constructions now give two monads on $\mathbf{DMod}(\mathbf{X})$: one by $(f_*, f^!)$ adjunction and one from the groupoid structure. The base change isomorphism

$$p_{t,*}p_s^! \to f^! f_*$$

gives an identification of these monads and hence of the algebras that they induce, i.e. for any $\mathscr{F} \in \mathbf{DMod}(\mathbf{X})$ we have

$$\mathrm{Hom}(\mathscr{F}, p_{t,*}p_s^!\mathscr{F}) \cong \mathrm{Hom}(\mathscr{F}, f^! f_* \mathscr{F}) \cong \mathrm{Hom}(f_*\mathscr{F}, f_*\mathscr{F}).$$

We will need to apply this construction for non-proper f. Unfortunately, in this case none of the adjunctions used to define the monads are available. We rectify this by restricting to the full subcategory of of holonomic D-modules and using the !-pushforward functors instead of the $*$-pushforward ones. Of course, by doing so we do not automatically have base change isomorphisms available anymore. Thus we have to explicitly require that all necessary base changes hold (this is usually called the Beck-Chevalley condition).

In order to formulate the condition, we need the concept of the *nerve* of a groupoid. This is the simplicial stack, also denoted $\mathbf{G}_\bullet$, with

$$\mathbf{G}_i = \underbrace{\mathbf{G}_1 \times_{\mathbf{G}_0} \cdots \times_{\mathbf{G}_0} \mathbf{G}_1}_{i \text{ factors}}.$$

We refer to [L1, Section 6.1.2] for the correct ∞-categorical setup. The following lemma is now an immediate corollary of [GR1, Lemma II.1.7.1.4] or [L2, Theorem 4.7.6.2].

Lemma 4.11. *Let $f : \mathbf{X} \to \mathbf{Y}$ be a schematic morphism of stacks and let $\mathbf{G}_\bullet$ be the corresponding groupoid. For each map $F : [n] \to [m]$ in $\mathbf{\Delta}^{\mathrm{op}}$ consider the corresponding square*

$$\begin{tikzcd} \mathbf{G}_{n+1} \arrow[r, "p_s"] \arrow[d, "p_{F+1}"] & \mathbf{G}_n \arrow[d, "p_F"] \\ \mathbf{G}_{m+1} \arrow[r, "p_s"] & \mathbf{G}_m \end{tikzcd}$$

where the vertical arrows are induced by F. Assume that for each such square the base change composition

$$p_{F+1,!}p_s^! \to p_{F+1,!}p_s^! p_F^! p_{F,!} = p_{F+1,!}p_{F+1}^! p_s^! p_{F,!} \to p_s^! p_{F,!}$$

given by the adjunction morphisms is an isomorphism of functors $\mathbf{DMod}_{\mathrm{hol}}(\mathbf{G}_n) \to \mathbf{DMod}_{\mathrm{hol}}(\mathbf{G}_{m+1})$. Then the endofunctor $p_{t,!}p_s^!$ of $\mathbf{DMod}_{\mathrm{hol}}(\mathbf{X})$ has a canonical structure of a monad and as such is isomorphic to the adjunction monad $f^! f_!$.

4.3. HOCHSCHILD COHOMOLOGY

We recall that the Hochschild cohomology of a dg category $\mathbf{C}$ is the algebra of derived endomorphisms of the identity functor,

$$\mathrm{HH}^\bullet(\mathbf{C}) = \mathbb{R}\mathrm{Hom}(\mathrm{Id}_{\mathbf{C}}, \mathrm{Id}_{\mathbf{C}}).$$

For the exact definition of the category $\mathbb{R}\mathrm{Hom} = \mathbf{Funct}(\mathbf{C}, \mathbf{C})$ we refer to [K2]. Instead we will give a more concrete construction via kernels which can be applied to $\mathbf{DMod}(\mathbf{X})$. For this let us restrict to the case of co-complete dg categories and let $\mathbf{Funct}_{\mathrm{cont}}(\mathbf{C}, \mathbf{C})$ be the full subcategory of $\mathbf{Funct}(\mathbf{C}, \mathbf{C})$ spanned by the continuous functors. Then, since $\mathrm{Id}_{\mathbf{C}}$ is evidently continuous, we have

$$\mathrm{HH}^\bullet(\mathbf{C}) = \mathrm{Hom}_{\mathbf{Funct}_{\mathrm{cont}}(\mathbf{C},\mathbf{C})}(\mathrm{Id}_{\mathbf{C}}, \mathrm{Id}_{\mathbf{C}}).$$

Let us further assume that $\mathbf{C}$ is dualizable with dual $\mathbf{C}^\vee$. Thus there is a unit map

$$\eta\colon \mathbf{Vect} \to \mathbf{C}^\vee \otimes \mathbf{C}$$

and a counit map

$$\varepsilon\colon \mathbf{C}^\vee \otimes \mathbf{C} \to \mathbf{Vect}$$

fulfilling the usual compatibilities (cf. [BN, Section 2]). Let $u = \eta(k)$. Then to each continuous endofunctor F of $\mathbf{C}$ we can associate its kernel $\mathrm{Id}_{\mathbf{C}^\vee} \otimes F(u) \in \mathbf{C}^\vee \otimes \mathbf{C}$ and conversely to each kernel $Q \in \mathbf{C}^\vee \otimes \mathbf{C}$ we can associate the endofunctor

$$\mathbf{C} \xrightarrow{\mathrm{Id}_{\mathbf{C}} \otimes Q} \mathbf{C} \otimes \mathbf{C}^\vee \otimes \mathbf{C} \xrightarrow{\varepsilon \otimes \mathrm{Id}_{\mathbf{C}}} \mathbf{C}.$$

These assignments are mutually inverse and give an equivalence of dg categories

$$\mathbf{Funct}_{\mathrm{cont}}(\mathbf{C}, \mathbf{C}) \cong \mathbf{C}^\vee \otimes \mathbf{C}.$$

In particular, the kernel for the identity is u and hence we have

$$\mathrm{HH}^\bullet(\mathbf{C}) = \mathrm{Hom}_{\mathbf{C}^\vee \otimes \mathbf{C}}(u, u).$$

Let us now consider the case of $\mathbf{C} = \mathbf{DMod}(\mathbf{X})$ for a stack X. Let $p\colon \mathbf{X} \to \mathrm{pt}$ be the structure morphism and $\Delta\colon \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ the diagonal. By [DG1, Section 8.4] the category $\mathbf{DMod}(\mathbf{X})$ is dualizable and there is a canonical indentification

$$\mathbf{DMod}(\mathbf{X})^\vee \otimes \mathbf{DMod}(\mathbf{X}) \cong \mathbf{DMod}(\mathbf{X} \times \mathbf{X})$$

such that the unit map is given by $\Delta_* p^!$ and thus we have

$$u = \Delta_* \omega_{\mathbf{X}}.$$

We summarize the above discussion in the following lemma.

Lemma 4.12. *Let* $\mathbf{X}$ *be a stack. Then the Hochschild cohomology of* $\mathbf{DMod}(\mathbf{X})$ *is given by the dg algebra*

$$\mathrm{HH}^\bullet(\mathbf{DMod}(\mathbf{X})) = \mathrm{Hom}_{\mathbf{DMod}(\mathbf{X}\times\mathbf{X})}(\Delta_* \omega_{\mathbf{X}}, \Delta_* \omega_{\mathbf{X}}).$$

5

BASE CHANGE FOR NON-PROPER MAPS

Let $\mathbf{X}$ be any stack, where we recall that all stacks are assumed to be QCA. We are interested in computing the Hochschild cohomology

$$\mathrm{HH}^\bullet(\mathbf{DMod}(\mathbf{X})).$$

Let $\Delta\colon \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ is the diagonal map, which by assumption is schematic. The dualizing module $\omega_\mathbf{X}$ is always holonomic. Thus we have $\mathbb{D}\Delta_*\omega_\mathbf{X} = \Delta_! k_\mathbf{X}$. With this we observe that

$$\begin{aligned}
\mathrm{HH}^\bullet(\mathbf{DMod}(\mathbf{X})) &= \mathrm{Hom}_{\mathbf{DMod}(\mathbf{X}\times\mathbf{X})}(\Delta_*\omega_\mathbf{X}, \Delta_*\omega_\mathbf{X}) && \text{(Lemma 4.12)}\\
&= \mathrm{Hom}_{\mathbf{DMod}(\mathbf{X}\times\mathbf{X})}(\Delta_! k_\mathbf{X}, \Delta_! k_\mathbf{X})^{\mathrm{op}} && \text{(duality)}\\
&= \mathrm{Hom}_{\mathbf{DMod}(\mathbf{X}\times\mathbf{X})}(k_\mathbf{X}, \Delta^!\Delta_! k_\mathbf{X})^{\mathrm{op}} && \text{(adjunction)}\\
&= \Gamma_{\mathrm{dR}}(\mathbf{X}, \Delta^!\Delta_! k_\mathbf{X})^{\mathrm{op}},
\end{aligned}$$

where the algebra structure on $\Gamma_{\mathrm{dR}}(\mathbf{X}, \Delta^!\Delta_! k_\mathbf{X}) = \mathrm{Hom}_{\mathbf{DMod}(\mathbf{X}\times\mathbf{X})}(k_\mathbf{X}, \Delta^!\Delta_! k_\mathbf{X})$ is the one induced by the $(\Delta_!, \Delta^!)$-adjunction monad. Consider the Cartesian square

$$\begin{tikzcd}
\mathscr{L}\mathbf{X} \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathbf{X} \arrow[d, "\Delta"] \\
\mathbf{X} \arrow[r, "\Delta"] & \mathbf{X} \times \mathbf{X}
\end{tikzcd}$$

Let us assume for the moment that Δ (and hence p_i) was proper. Then $\Delta_* = \Delta_!$ and $p_{2,*} = p_{2,!}$ and by Section 4.2 we have an isomorphism of monads

$$p_{2,!}p_1^! \to \Delta^!\Delta_!, \tag{5.1}$$

which induces an isomorphism of algebras

$$\Gamma_{\mathrm{dR}}(\mathbf{X}, p_{2,!}p_1^! k_\mathbf{X}) \to \Gamma_{\mathrm{dR}}(\mathbf{X}, \Delta^!\Delta_! k_\mathbf{X}).$$

Of course, if X is not an algebraic space, then Δ is not proper (nor is it in general smooth). Thus in general (5.1) is not an isomorphism and there is no canonical structure of monad on $p_{2,!}p_1^!$. We would like to apply Lemma 4.11 to construct a monad in special cases. Thus the goal of this chapter is to give a criterion for the assumptions of Lemma 4.11, i.e. for base change to hold.

Example 5.1. The base change morphism (5.1) is also an isomorphism if Δ is smooth. In particular this implies that the "naive expectation" holds for $\mathbf{X} = \mathbf{B}G$ for any algebraic group G, i.e. we have

$$\mathrm{HH}^\bullet(\mathbf{DMod}(\mathbf{B}G)) = \Gamma_{\mathrm{dR}}(\mathbf{B}G, p_{2,!}p_1^! k_{\mathbf{B}G})^{\mathrm{op}}.$$

An argument similar to [B1] shows that there is a further isomorphism

$$\Gamma_{\mathrm{dR}}(\mathbf{B}G, p_{2,!}p_1^! k_{\mathbf{X}}) \cong \Gamma_{\mathrm{dR}}(G, k_G)^\vee \otimes \Gamma_{\mathrm{dR}}(\mathbf{B}G, k_{\mathbf{B}G}).$$

Alternatively, we can use the identification

$$\mathbf{DMod}(\mathbf{B}G) \cong \Gamma_{\mathrm{dR}}(G, k_G)^\vee\text{-}\mathbf{Mod},$$

where the algebra structure on $\Gamma_{\mathrm{dR}}(G, k_G)^\vee$ is induced by the group multiplication [DG1, Section 7.2]. If G is reductive, then $\Gamma_{\mathrm{dR}}(G, k_G)^\vee$ is an exterior algebra and thus its Hochschild cohomology can be computed directly. ○

5.1. A LEMMA ON BASE CHANGE

Consider a Cartesian diagram of stacks

$$\begin{tikzcd} \mathbf{Z} \arrow[r, "q"] \arrow[d, "p"] & \mathbf{X}_1 \arrow[d, "f"] \\ \mathbf{X}_2 \arrow[r, "g"] & \mathbf{Y} \end{tikzcd}$$

with f and g schematic. We have a morphism of functors $\mathbf{DMod}_{\mathrm{hol}}(\mathbf{X}_1) \to \mathbf{DMod}_{\mathrm{hol}}(\mathbf{X}_2)$,

$$p_! q^! \to g^! f_! \tag{5.2}$$

induced by adjunctions

$$p_! q^! \to p_! q^! f^! f_! = p_! p^! g^! f_! \to g^! f_!. \tag{5.3}$$

If f is proper, then (5.2) is an isomorphism by Proposition 4.4. To understand the behavior for non-proper f, we will approximate it by a proper morphism.

Definition 5.2. A *relative compactification* of a morphism $f : \mathbf{X} \to \mathbf{Y}$ is a commutative diagram

$$\begin{tikzcd} \mathbf{X} \arrow[r, hook, "j"] \arrow[dr, "f"'] & \overline{\mathbf{X}} \arrow[d, "\overline{f}"] \\ & \mathbf{Y} \end{tikzcd}$$

where j is an open embedding and $\overline{f}$ is proper.

Example 5.3. A famous example of such a relative compactification is Drinfeld's compactification of the morphism $\mathrm{Bun}_B \to \mathrm{Bun}_G$, where Bun_G is the stack of G-bundles on a curve C with G reductive and B is a Borel subgroup of G [BG]. ○

Let us assume that in the above situation there exists a relative compactification of $f : \mathbf{X}_1 \to \mathbf{Y}$. Let $\mathbf{X}_1^c$ be the closed complement of the open inclusion $j : \mathbf{X}_1 \hookrightarrow \overline{\mathbf{X}}_1$. Similarly, we let $\overline{\mathbf{Z}} = \mathbf{X}_2 \times_{\mathbf{Y}} \overline{\mathbf{X}}_1$ and $\mathbf{Z}^c = \mathbf{X}_2 \times_{\mathbf{Y}} \mathbf{X}_1^c$. The notation for the corresponding inclusion and projection maps is summarized in the following Cartesian diagrams.

$$\begin{tikzcd} \overline{\mathbf{Z}} \arrow[r, "\overline{q}"] \arrow[d, "\overline{p}"] & \overline{\mathbf{X}}_1 \arrow[d, "\overline{f}"] \\ \mathbf{X}_2 \arrow[r, "g"] & \mathbf{Y} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Z}^c \arrow[r, hook, "i"] \arrow[d] & \overline{\mathbf{Z}} \arrow[d, "\overline{q}"] \\ \mathbf{X}_1^c \arrow[r, hook] & \overline{\mathbf{X}}_1 \end{tikzcd}$$

We note that $\overline{\mathbf{Z}}$ is the disjoint union of the closed substack $\mathbf{Z}^c$ and the open substack $\mathbf{Z}$.

Lemma 5.4. *The cone of the morphism* (5.2) *is*

$$\overline{p}_! i_* i^* \overline{q}^! j_!.$$

In particular, if $i^*\overline{q}^! j_! = 0$, *then* (5.2) *is an isomorphism of functors.*

Proof. Let $\tilde{j} : \mathbf{Z} \hookrightarrow \overline{\mathbf{Z}}$ be the open inclusion complement to i. We split the adjunction in (5.3) in two by using the compositions

$$f = \overline{f} \circ j,\ p = \overline{p} \circ \tilde{j} \text{ and } q = \overline{q} \circ \tilde{j}.$$

Thus the adjunction $p_! q^! \to p_! q^! f^! f_!$ becomes the sequence

$$p_! q^! \to p_! q^! j^! j_! \to p_! q^! j^! \overline{f}^! \overline{f}_! j_!.$$

The equality $p_! q^! f^! f_! = p_! p^! g^! f_!$ then becomes

$$p_! q^! j^! \overline{f}^! \overline{f}_! j_! = p_! \tilde{j}^! \overline{q}^! \overline{f}^! \overline{f}_! j_! = p_! \tilde{j}^! \overline{p}^! g^! \overline{f}_! j_!.$$

Finally the adjunction $p_! p^! g^! f_! \to g^! f_!$ becomes

$$p_! \tilde{j}^! \overline{p}^! g^! \overline{f}_! j_! = \overline{p}_! \tilde{j}_! \tilde{j}^! \overline{p}^! g^! \overline{f}_! j_! \to \overline{p}_! \overline{p}^! g^! \overline{f}_! j_! \to g^! \overline{f}_! j_! = g^! f_!.$$

Let us apply the same adjunction morphisms in a different order. First the inclusions

$$p_! q^! \xrightarrow{\alpha} p_! q^! j^! j_! = \overline{p}_! \tilde{j}_! \tilde{j}^! \overline{q}^! j_! \xrightarrow{\beta} \overline{p}_! \overline{q}^! j_!,$$

and then the actual base change

$$\overline{p}_! \overline{q}^! j_! \to \overline{p}_! \overline{q}^! \overline{f}^! \overline{f}_! j_! = \overline{p}_! \overline{p}^! g^! \overline{f}_! j_! \to g^! \overline{f}_! j_! = g^! f_!. \tag{5.4}$$

We note that the adjunction $\alpha\colon \mathrm{Id} \to j^! j_!$ is an isomorphism and the composition of the maps in (5.4) is exactly the isomorphism of proper base change (cf. Proposition 4.4). Thus the cone of the whole composition is the same as the cone of the morphism β, which is given by the recollement triangle

$$\overline{p}_! \tilde{j}_! \tilde{j}^! \overline{q}^! j_! \xrightarrow{\beta} \overline{p}_! \overline{q}^! j_! \longrightarrow \overline{p}_! i_* i^* \overline{q}^! j_! \xrightarrow{+1} . \qquad \square$$

5.2. RELATIVE COMPACTIFICATION FOR QUOTIENT STACKS

In the preceding section we simply assumed that a relative compactification of the diagonal exists. We will now construct such a compactification for quotient stacks. Thus let X be a scheme of finite type over k and let G be an affine algebraic group over k acting on X. Let $\mathbf{X} = X/G$ be the corresponding quotient stack.

Constructing a relative compactification of $\Delta\colon \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ is the same as first constructing a $G \times G$-equivariant relative compactification of $(\mathrm{pr}_2, a)\colon G \times X \to X \times X$ (where $a\colon G \times X \to X$ is the action map) and then taking the quotient by the $G \times G$ action[1]. We let

$$\Gamma = \{(g, x, x, gx) \in G \times X \times X \times X\}$$

be the graph of (pr_2, a).

We pick a $G \times G$-equivariant compactification $\overline{G}$ of G and let $\overline{\Gamma}$ be the closure of Γ in $\overline{G} \times X \times X \times X$. We have an open embedding j of $G \times X \cong \Gamma$ into $\overline{\Gamma}$ and proper map $f\colon \overline{\Gamma} \to X \times X$ given by projection on the last two factors. The composition $f \circ j$ is equal to (pr_2, a).

Instead of viewing Γ as the graph of (pr_2, a) we can drop the third factor and regard Γ as the graph of the action map, i.e.

$$\Gamma_a = \{(g, x, gx) \in G \times X \times X\}.$$

The closure $\overline{\Gamma_a}$ of Γ_a in $\overline{G} \times X \times X$ identifies with $\overline{\Gamma}$. Thus for ease of notation we will from now on always set $\Gamma = \Gamma_a$ and $\overline{\Gamma} = \overline{\Gamma_a}$.

Definition 5.5. Let $\mathbf{X} = X/G$. With the above construction we set

$$\overline{\mathbf{X}} = \overline{\Gamma}\big/_{G \times G} .$$

We have an open embedding $j\colon X \hookrightarrow \overline{\mathbf{X}}$ and a proper morphism $\overline{\Delta}\colon \overline{\mathbf{X}} \to \mathbf{X} \times \mathbf{X}$ induced by the map f above, such that $\Delta = \overline{\Delta} \circ j$.

Remark 5.6. In the case of $G = \mathbb{G}_m$ the compactification $\overline{\Gamma}$ is explicitly described in [DG2]. In particular, if X is smooth it is shown there that $\overline{\Gamma}$ is smooth over $\overline{G} = \mathbb{P}^1$. It is possible to extend the methods of [DG2] to quotients by higher dimensional tori. The resulting constructions are highly useful for doing explicit computations.

[1] Here $G \times G$ acts on $G \times X$ by $(s_1, s_2) \cdot (g, x) = (s_2 g s_1^{-1}, s_1 x)$.

It is useful to consider only partial compactifications. For this let V be a G-stable subvariety of $\overline{G}$ and let $\overline{\Gamma_V}$ be the closure of Γ in $G \times V$. We set

$$\overline{\mathbf{X}}_V = \overline{\Gamma_V}\big/G \times G\,.$$

Clearly, if $\{V_i\}$ is an open cover of $\overline{G}$ by G-stable subvarieties, then $\{\overline{\mathbf{X}}_{V_i}\}$ is an open cover of $\overline{\mathbf{X}}$.

5.3. GOOD STACKS

Let $\mathbf{X} = X/G$ be a quotient stack as before. For any morphism of stacks $h\colon \mathbf{Y} \to \mathbf{X}$ we set $\mathscr{L}_{\mathbf{Y}}\mathbf{X} = \mathbf{X} \times_{\mathbf{X}\times\mathbf{X}} \mathbf{Y}$. Thus we have the Cartesian diagram

$$\begin{tikzcd} \mathscr{L}_{\mathbf{Y}}\mathbf{X} \arrow[r, "q_{\mathbf{Y}}"] \arrow[d] & \mathbf{X} \arrow[d, "\Delta"] \\ \mathbf{Y} \arrow[r, "\Delta\circ h"] & \mathbf{X}\times\mathbf{X} \end{tikzcd}$$

Let us fix a relative compactification $\overline{\Delta}\colon \overline{\mathbf{X}} \to \mathbf{X}\times\mathbf{X}$ as in Section 5.2. Using the notation of Section 5.1 we set $\overline{\mathscr{L}_{\mathbf{Y}}\mathbf{X}} = \overline{\mathbf{X}} \times_{\mathbf{X}\times\mathbf{X}} \mathbf{Y}$ and $\mathscr{L}^c_{\mathbf{Y}}\mathbf{X} = \mathbf{X}^c \times_{\mathbf{X}\times\mathbf{X}} \mathbf{Y}$. We let $\overline{q}_{\mathbf{Y}}\colon \overline{\mathscr{L}_{\mathbf{Y}}\mathbf{X}} \to \overline{\mathbf{X}}$ be the projection morphism and $i_{\mathbf{Y}}\colon \mathscr{L}^c_{\mathbf{Y}}\mathbf{X} \hookrightarrow \overline{\mathscr{L}_{\mathbf{Y}}\mathbf{X}}$ the inclusion. Thus we have the following central diagram

$$\begin{tikzcd} \mathscr{L}_{\mathbf{Y}}\mathbf{X} \arrow[r, hook] \arrow[d, "q_{\mathbf{Y}}"] & \overline{\mathscr{L}_{\mathbf{Y}}\mathbf{X}} \arrow[d, "\overline{q}_{\mathbf{Y}}"] & \mathscr{L}^c_{\mathbf{Y}}\mathbf{X} \arrow[l, hook', "i_{\mathbf{Y}}"'] \arrow[d] \\ \mathbf{X} \arrow[r, hook, "j"] & \overline{\mathbf{X}} & \mathbf{X}^c \arrow[l, hook'] \end{tikzcd}$$

Definition 5.7. A quotient stack $\mathbf{X} = X/G$ is called *good* if for every quotient stack $\mathbf{Y} = Y/G$ and schematic morphism $\mathbf{Y} \to \mathbf{X}$ the functor $i^*_{\mathbf{Y}}\overline{q}^!_{\mathbf{Y}}j_!$ vanishes on $\mathbf{DMod}_{\mathrm{hol}}(\mathbf{X})$.

We will show in Chapter 6 that any stack of the form $X/\mathbb{G}^n_m$ is good. The reason for this definition is the following theorem which lets us compute the Hochschild cohomology of $\mathbf{DMod}(\mathbf{X})$ for good quotient stacks.

Theorem 5.8. *If $\mathbf{X} = X/G$ is good, then there exists a canonical structure of monad on $p_{2,!}p^!_1$ and the morphism $p_{2,!}p^!_1 \to \Delta^!\Delta_!$ is an isomorphism of monads. In particular there is an isomorphism of algebras*

$$\mathrm{HH}^\bullet(\mathbf{DMod}(\mathbf{X})) \cong \Gamma_{\mathrm{dR}}(\mathbf{X}, p_{2,!}p^!_1 k_{\mathbf{X}})^{\mathrm{op}}.$$

In other words, Theorem 1.2 holds for good stacks.

Proof. We apply Lemma 4.11 to the groupoid $\mathscr{L}\mathbf{X} \rightrightarrows \mathbf{X}$. Thus we let $\mathbf{G}_\bullet$ be the simplicial stack with

$$\mathbf{G}_i = \underbrace{\mathscr{L}\mathbf{X} \times_{\mathbf{X}} \cdots \times_{\mathbf{X}} \mathscr{L}\mathbf{X}}_{i \text{ factors}}.$$

Any morphism $F\colon [n] \to [m]$ in $\mathbf{\Delta}^{\mathrm{op}}$ induces a diagram

$$\begin{tikzcd}
\mathbf{G}_{n+1} \arrow[r] \arrow[d] & \mathbf{G}_{m+1} \arrow[r] \arrow[d] & \mathbf{X} \arrow[d, "\Delta"] \\
\mathbf{G}_n \arrow[r] & \mathbf{G}_m \arrow[r] & \mathbf{X} \times \mathbf{X}
\end{tikzcd}$$

We have to show that base change holds along the left-hand square. But by the assumption and Lemma 5.4, base change holds along the outer rectangle and the left-hand square. Thus it also holds along the left-hand square. ☐

Remark 5.9. We expect that most quotient stacks are not good. For example, a direct computation shows that Theorem 1.2 does not hold for the stack $\mathbb{P}^1/\mathbb{A}^1$, and hence it is not good. For non-good stacks, Lemma 5.4 instead gives a description of how much the naive expectation for $\mathrm{HH}^\bullet(\mathbf{DMod}(\mathbf{X}))$ fails.

We finish this section with some useful observations for proving that a stack is good.

Lemma 5.10. *If $\mathbf{X}_1 = X_1/G_1$ and $\mathbf{X}_2 = X_2/G_2$ are good, then $\mathbf{X}_1 \times \mathbf{X}_2$ is good.*

Proof. Follows from compatibility of $i^*_{\mathbf{Y}}\overline{q}^!_{\mathbf{Y}}j_!$ with $\boxtimes$ and coproducts. (Note that $\mathscr{L}^c_{\mathbf{Y}}(\mathbf{X}_1 \times \mathbf{X}_2) = \mathscr{L}^c_{\mathbf{Y}}\mathbf{X_1} \times \overline{\mathscr{L}}_{\mathbf{Y}}\mathbf{X_2} \cup \overline{\mathscr{L}}_{\mathbf{Y}}\mathbf{X}_1 \times \mathscr{L}^c_{\mathbf{Y}}\mathbf{X}_2$.) ☐

Lemma 5.11. *Let U be a G-equivariant open subset of X. If X/G is good then U/G is good.*

Proof. Let $\mathbf{U} = U/G$ and let $\mathbf{Y}$ be a quotient stack mapping into $\mathbf{U}$ (and hence also into $\mathbf{X}$). Consider the diagram

$$\begin{tikzcd}
\mathscr{L}^c_{\mathbf{Y}}\mathbf{U} \arrow[r, hook, "i_{\mathbf{U},\mathbf{Y}}"] \arrow[d, hook, "\alpha"] & \overline{\mathscr{L}}_{\mathbf{Y}}\mathbf{U} \arrow[r, "\overline{q}_{\mathbf{U},\mathbf{Y}}"] \arrow[d, hook, "\beta"] & \overline{\mathbf{U}} \arrow[d, hook, "\gamma"] & \mathbf{U} \arrow[l, hook', "j_{\mathbf{U}}"'] \arrow[d, hook, "\delta"] \\
\mathscr{L}^c_{\mathbf{Y}}\mathbf{X} \arrow[r, hook, "i_{\mathbf{X},\mathbf{Y}}"] & \overline{\mathscr{L}}_{\mathbf{Y}}\mathbf{X} \arrow[r, "\overline{q}_{\mathbf{X},\mathbf{Y}}"] & \overline{\mathbf{X}} & \mathbf{X} \arrow[l, hook', "j_{\mathbf{X}}"']
\end{tikzcd} \tag{5.5}$$

The vertical arrows are open embeddings and all squares are Cartesian (where we use the same compactification of G for $\overline{\mathbf{X}}$ and $\overline{\mathbf{U}}$). Thus

$$i^*_{\mathbf{U},\mathbf{Y}}\overline{q}^!_{\mathbf{U},\mathbf{Y}}j_{\mathbf{U},!} = i^*_{\mathbf{U},\mathbf{Y}}\overline{q}^!_{\mathbf{U},\mathbf{Y}}j_{\mathbf{U},!}\delta^*\delta_* = \alpha^* i^*_{\mathbf{X},\mathbf{Y}}\overline{q}^!_{\mathbf{X},\mathbf{Y}}j_{\mathbf{X},!}\delta_* = 0.$$ ☐

The same argument can be used to reduce the computation to a smooth cover. We will now introduce notation for the special case of the cover $\overline{\Gamma} \to \overline{\mathbf{X}}$. The corresponding covers of the other relevant stacks are introduced in the following diagram with Cartesian squares.

$$\begin{tikzcd} \mathscr{I}^c_{\mathbf{Y}}\mathbf{X} \arrow[r, hook, "u_{\mathbf{Y}}"] \arrow[d] & \overline{\mathscr{I}}_{\mathbf{Y}}\mathbf{X} \arrow[r, "\overline{\kappa}_{\mathbf{Y}}"] \arrow[d] & \overline{\Gamma} \arrow[d] & \Gamma \arrow[l, hook', "\breve{u}"'] \arrow[d] \\ \mathscr{L}^c_{\mathbf{Y}}\mathbf{X} \arrow[r, hook, "i_{\mathbf{Y}}"] & \overline{\mathscr{L}}_{\mathbf{Y}}\mathbf{X} \arrow[r, "\overline{q}_{\mathbf{Y}}"] & \overline{\mathbf{X}} & \mathbf{X} \arrow[l, hook', "j"'] \end{tikzcd} \tag{5.6}$$

We note that all vertical morphisms are smooth and the spaces in the top row are schemes. Let $h' : X \to Y$ be the G-equivariant morphism of schemes inducing h on quotient stacks. Then the scheme $\overline{\mathscr{I}}_{\mathbf{Y}}\mathbf{X}$ is given by

$$\overline{\mathscr{I}}_{\mathbf{Y}}\mathbf{X} = \left\{(g_1, y, g_2) \in G \times Y \times \overline{G} : (g_2, h'(y), g_1 h'(y)) \in \overline{\Gamma}\right\}.$$

Lemma 5.12. *A stack X/G is good if and only if for each morphism $Y/G \to X/G$ the composition $u^*_{\mathbf{Y}}\overline{\kappa}^!_{\mathbf{Y}}\breve{u}_!$ vanishes on $\mathbf{DMod}_{\mathrm{hol}}(\Gamma)^{G\times G-\mathrm{mon}}$.*

Proof. Follows from the fact that pullback along the smooth vertical morphisms in (5.6) is conservative [DG1, Lemma 5.1.6] and permutes with the other morphisms up to a shift. □

Lemma 5.13. *If there exists a G-stable open cover U_i of X such that all stacks U_i/G are good, then X/G is good.*

Proof. Let $\mathbf{U}_i = U_i/G$ be the corresponding quotient stacks. We first show that the stacks $\mathscr{L}^c_{\mathbf{Y}}\mathbf{U}_i$ form an open cover of $\mathscr{L}^c_{\mathbf{Y}}\mathbf{X}$[2]. For this it suffices to show that the open subschemes $\overline{\mathscr{I}}_{\mathbf{Y}}\mathbf{U}_i$ cover $\overline{\mathscr{I}}_{\mathbf{Y}}\mathbf{X}$. Let (g_1, y, g_2) be a point of $\overline{\mathscr{I}}_{\mathbf{Y}}\mathbf{X}$. Then there exists some U_i with $h'(y) \in U_i$. But then $g_1 h'(y)$ is also in U_i and hence $(h'(y), g_2, g_1 h'(y)) \in \overline{\mathbf{U}}_i$. Thus (g_1, y, g_2) is in $\overline{\mathscr{I}}_{\mathbf{Y}}\mathbf{U}_i$.

It now suffices to show that the restrictions of $i^*_{\mathbf{Y}}\overline{q}^!_{\mathbf{Y}}j_!\mathscr{F}$ to $\mathscr{L}^c_{\mathbf{Y}}\mathbf{U}_i$ vanish for every $\mathscr{F} \in \mathbf{DMod}(\mathbf{X})$. But this follows from the diagram (5.5) (for $\mathbf{U}_i$ instead of $\mathbf{U}$) and the goodness of $\mathbf{U}_i$. □

Let $\{V_i\}$ be a G-stable open cover of $\overline{G}$ and consider the corresponding open cover $\{\overline{\mathbf{X}}_{V_i}\}$ of $\mathbf{X}$. We obtain open covers $\{\overline{\mathscr{L}}_{V_i,\mathbf{Y}}\mathbf{X}\}$ and $\{\mathscr{L}^c_{V_i,\mathbf{Y}}\mathbf{X}\}$ of $\overline{\mathscr{L}}_{\mathbf{Y}}\mathbf{X}$ and $\mathscr{L}^c_{\mathbf{Y}}\mathbf{X}$ respectively. We let $i_{V_i,\mathbf{X}}$, $\overline{q}_{V_i,\mathbf{X}}$ and j_{V_i} be the corresponding maps, i.e.

$$\overline{\mathscr{L}}_{V_i,\mathbf{Y}}\mathbf{X} \xhookrightarrow{i_{V_i,\mathbf{X}}} \mathscr{L}^c_{V_i,\mathbf{Y}}\mathbf{X} \xrightarrow{\overline{q}_{V_i,\mathbf{X}}} \overline{\mathbf{X}}_{V_i} \xhookleftarrow{j_{V_i}} \mathbf{X}.$$

Lemma 5.14. *With the above notation, the $\mathbf{X}$ is good if and only $i^*_{V_i,\mathbf{Y}}\overline{q}^!_{V_i,\mathbf{Y}}j_{V_i,!}$ vanishes on $\mathbf{DMod}_{\mathrm{hol}}(\mathbf{X})$ for all V_i and all $\mathbf{Y} \to \mathbf{X}$.*

[2] This is not completely obvious, since the $\overline{\mathbf{U}}_i$ do not necessarily form a cover of $\overline{\mathbf{X}}$. For example, consider $\mathbb{P}^1$ with the usual linear $\mathbb{G}_m$-action and the usual affine cover.

Proof. Similar to the proof of Lemma 5.13. ∎

6

TORUS QUOTIENTS

In this chapter we will apply the tools from the previous chapter to torus quotient stacks. Specifically, we will prove the following theorem.

Theorem 6.1. *Let $G \cong \mathbb{G}_m^n$ be a torus acting locally linearly on a scheme X of finite type over k. Then the stack $\mathbf{X} = X/G$ is good.*

Together with Theorem 5.8 this implies our main result, Theorem 1.2.

Remark 6.2. We only use the assumption that the action is locally linear to prove Lemma 6.4, i.e. that $\operatorname{Stab}\mathbf{X}$ is locally finite. Thus it would suffice to assume that X can be covered by open subschemes for which Lemma 6.4 holds.

Remark 6.3. Theorem 6.1 actually holds for G the product of a torus and a finite Abelian group. The argument is exactly the same.

By Lemma 5.13, it suffices to prove Theorem 6.1 for stacks X/G with X affine. We fix an isomorphism $G \cong \mathbb{G}_m^n$ and compactify G to $(\mathbb{P}^1)^n$. The variety $(\mathbb{P}^1)^n$ can be covered by G-equivariant open subvarieties of the form $\mathbb{A}^n$. Thus by Lemma 5.14, it suffices to check goodness for the relative compactification $\mathbb{G}_m^n \subseteq \mathbb{A}^n$. To simplify notation, we drop the subscript $\mathbb{A}^n$ from the notation and set $\overline{\mathbf{X}} = \overline{\mathbf{X}}_{\mathbb{A}^n}$ and similarly for the various maps.

We fix a quotient stack $\mathbf{Y} = Y/G$ and a morphism $h\colon \mathbf{Y} \to \mathbf{X}$, induced by a G-equivariant morphism $h'\colon Y \to X$. According to Lemma 5.12, rather than working with stack directly, we can base change to schemes. We will use the notation of Lemma 5.12, but for ease of notation we will drop the subscript $\mathbf{Y}$ from the maps. Thus we are concerned with the diagram

$$\mathit{\Pi}^c_{\mathbf{Y}}\mathbf{X} \xhookrightarrow{u} \overline{\mathit{\Pi}}_{\mathbf{Y}}\mathbf{X} \xrightarrow{\overline{\kappa}} \overline{\Gamma} \xhookleftarrow{\breve{u}} \Gamma,$$

where we have to show that $u^*\overline{\kappa}^!\breve{u}_!$ vanishes on $\mathbf{DMod}_{\mathrm{hol}}(\Gamma)^{G\times G-\mathrm{mon}}$.

The general idea is to introduce a $\mathbb{G}_m$-action that contracts $\mathit{\Pi}^c_{\mathbf{Y}}\mathbf{X}$ onto $\overline{\mathit{\Pi}}_{\mathbf{Y}}\mathbf{X}$. In order to do so, we will cut the scheme

$$\overline{\mathit{\Pi}}_{\mathbf{Y}}\mathbf{X} = \left\{(g_1, y, g_2) \in G \times Y \times \overline{G} : (g_2, h'(y), g_1 h'(y)) \in \overline{\Gamma}\right\}$$

into pieces according to the subgroups of G that stabilize $h'(y)$. For this let $\operatorname{Stab}\mathbf{X}$ be the set of all closed subgroups of G that are stabilizers of points of X, i.e.

$$\operatorname{Stab}\mathbf{X} = \{G_x : x \in X\}.$$

Lemma 6.4. *The set* $\operatorname{Stab}\mathbf{X}$ *is finite.*

Proof. Since X is affine, it can be embedded G-equivariantly into some $\mathbb{A}^m$ with a linear T-action. For $\mathbb{A}^m/T$ the statement is easy to see. $\square$

Let S be closed subgroup of G and let X^S be the S-fixed points of X. Since G is Abelian (and hence S a normal subgroup), X^S is a G-stable closed subscheme of X. Hence X^S/G is a closed substack of $\mathbf{X}$. Let $\overline{S}$ be the closure of S in $\overline{G} = \mathbb{A}^n$ and consider the space

$$\overline{Л}^S_{\mathbf{Y}}\mathbf{X} = \Big\{(g_1, y, g_2) \in G\times Y\times\overline{G} : h'(y) \in X^S,\ (g_2, h'(y), g_1h'(y)) \in \overline{\Gamma} \text{ and } g_2 \in g_1\overline{S}\Big\} \subseteq \overline{Л}_{\mathbf{Y}}\mathbf{X}.$$

Lemma 6.5. *The subspaces* $\overline{Л}^S_{\mathbf{Y}}\mathbf{X}$ *for* $S \in \operatorname{Stab}\mathbf{X}$ *cover* $\overline{Л}_{\mathbf{Y}}\mathbf{X}$.

Proof. Let $Л_{\mathbf{Y}}\mathbf{X}$ be the smooth cover of $\mathscr{L}_{\mathbf{Y}}\mathbf{X}$. Consider the spaces

$$Л^S_{\mathbf{Y}}\mathbf{X} = \Big\{(g_1, y, g_2) \in G\times Y\times G : h'(y) \in X^S,\ (g_2, h'(y), g_1h'(y)) \in \Gamma \text{ and } g_2 \in g_1S\Big\} \subseteq Л_{\mathbf{Y}}\mathbf{X}.$$

The closure of $Л^S_{\mathbf{Y}}\mathbf{X}$ in $\overline{Л}_{\mathbf{Y}}\mathbf{X}$ is exactly $\overline{Л}^S_{\mathbf{Y}}\mathbf{X}$. It is easy to see that the subspaces $Л^S_{\mathbf{Y}}\mathbf{X}$ for $S \in \operatorname{Stab}\mathbf{X}$ cover $Л^S_{\mathbf{Y}}\mathbf{X}$. Now the statement follows from the fact that the closure of a finite union is the union of the individual closures. $\square$

It will be useful to have a slight generalization of the schemes $\overline{Л}^S_{\mathbf{Y}}\mathbf{X}$. Let $S_1 \subseteq S_2$ be two subgroups of G contained in $\operatorname{Stab}\mathbf{X}$. We set

$$\overline{Л}^{S_1,S_2}_{\mathbf{Y}}\mathbf{X} = \Big\{(g_1, y, g_2) \in G \times Y \times \overline{G} : h'(y) \in X^{S_2},\ (g_2, h'(y), g_1h'(y)) \in \overline{\Gamma} \text{ and } g_2 \in g_1\overline{S}_1\Big\}.$$

Clearly we have $\overline{Л}^{S_1,S_2}_{\mathbf{Y}}\mathbf{X} \subseteq \overline{Л}^{S_1}\mathbf{X}$ and $\overline{Л}^S_{\mathbf{Y}}\mathbf{X} = \overline{Л}^{S,S}_{\mathbf{Y}}\mathbf{X}$.

Consider the Cartesian square of closed embeddings

$$\begin{tikzcd}
Л^{S_1,S_2,c}_{\mathbf{Y}}\mathbf{X} \arrow[r, hook, "i^c_{S_1,S_2}"] \arrow[d, hook, "u^{S_1,S_2}"] & Л^c_{\mathbf{Y}}\mathbf{X} \arrow[d, hook, "u"] \\
\overline{Л}^{S_1,S_2}_{\mathbf{Y}}\mathbf{X} \arrow[r, hook, "i_{S_1,S_2}"] & \overline{Л}_{\mathbf{Y}}\mathbf{X}
\end{tikzcd}$$

Lemma 6.6. *For any* $S_1 \subseteq S_2$ *in* $\operatorname{Stab}\mathbf{X}$ *and any* $\mathscr{F} \in \mathbf{DMod}_{\mathrm{hol}}(\Gamma)^{G\times G-\mathrm{mon}}$ *we have*

$$u^{S_1,S_2,*}i^!_{S_1,S_2}\overline{\kappa}^!\check{u}_!\mathscr{F} = 0.$$

Proof. The scheme $Л^{S_1,S_2,c}\mathbf{X}$ is given by

$$\Big\{(g_1, y, g_2) \in G \times Y \times \overline{G} : h'(y) \in X^{S_2},\ (g_2, h'(y), g_1h'(y)) \in \overline{\Gamma} \text{ and } g_2 \in g_1(\overline{S}_1 - S_1)\Big\}.$$

If $S_1 = \overline{S}_1$ the statement is trivially true. Otherwise the scheme $\overline{S}_1 - S_1$ is the union of hyperplanes H_i of $\overline{S}_1$. It suffices to prove the statement when further restricting to

$$\left\{(g_1, y, g_2) \in G \times Y \times \overline{G} : h'(y) \in X^{S_2},\ (g_2, h'(y), g_1 h'(y)) \in \Gamma \text{ and } g_2 \in g_1 H_i\right\}.$$

for all i. Let H be one such hyperplane. We will assume that H is contained in the closure of the connected component of $1 \in S_1$. The proof for H in a different component is the same, up to a shift by an element of G. Let u_H be the inclusion of

$$Z = \left\{(g_1, y, g_2) \in G \times Y \times \overline{G} : h'(y) \in X^{S_2},\ (g_2, h'(y), g_1 h'(y)) \in \Gamma \text{ and } g_2 \in g_1 H\right\}$$

into $\overline{\mathit{\Pi}}^{S_1,S_2}_{\mathbf{Y}}\mathbf{X}$. We want to compute

$$u_H^* i^!_{S_1,S_2} \overline{\kappa}^! \check{u}_! \mathscr{F}.$$

We will do so be introducing a contractive $\mathbb{G}_m$-action on $\overline{\Gamma}$ and $\mathit{\Pi}^{S_1,S_2}_{\mathbf{Y}}\mathbf{X}$ such that the morphism

$$\kappa \circ i_{S_1,S_2} : \overline{\mathit{\Pi}}^{S_1,S_2}_{\mathbf{Y}}\mathbf{X} \to \overline{\Gamma}$$

is $\mathbb{G}_m$-equivariant.

We write $G \cong G_1 \times S_1$ for some subgroup G_1 of G. This given a corresponding decomposition of the monad $\overline{G} \cong \mathbb{A}^n$ as $\overline{G} = \overline{G}_1 \overline{S}_1$. Let $H' = \overline{G}_1 H$. We note that $G \cap H' = \emptyset$.

We chose an action μ of $\mathbb{G}_m$ on $\overline{S}_1$ that contracts $\overline{S}_1$ onto H. This induces an action of $\mathbb{G}_m$ on $\overline{G} = \overline{G}_1 \overline{S}_1$ by $u \cdot ts = t\mu(u,s)$, contracting $\overline{G}$ onto H'. Further we obtain an action of $\mathbb{G}_m$ on $\overline{\Gamma}$ that keeps the first X coordinate fixed. By construction this action contracts $\overline{\Gamma}$ onto a closed subvariety of $\overline{\Gamma} - \Gamma$. We will denote this subvariety by Z_1 and the contraction morphism $\pi_1 : \overline{\Gamma} \to Z_1$ by π_1.

We can also lift the action to $\overline{\mathit{\Pi}}^{S_1,S_2}_{\mathbf{Y}}\mathbf{X}$ where it contracts onto Z. We will denote the corresponding contraction morphism by $\pi : \overline{\mathit{\Pi}}^{S_1,S_2}_{\mathbf{Y}}\mathbf{X} \to Z$. The morphism $\kappa \circ i_{S_1,S_2} : \overline{\mathit{\Pi}}^{S_1,S_2}_{\mathbf{Y}}\mathbf{X} \to \overline{\Gamma}$ is equivariant with respect to these $\mathbb{G}_m$-actions and on its image the action keeps the second X-coordinate fixed.

We note that the D-modules $\check{u}_! \mathscr{F}$ and $i^!_{S_1,S_2} \overline{\kappa}^! \check{u}_! \mathscr{F}$ are monodromic with respect to these $\mathbb{G}_m$-actions. Thus the contraction principle Theorem 4.9 implies that

$$u_H^* i^!_{S_1,S_2} \overline{\kappa}^! \check{u}_! \mathscr{F} = \pi_* i^!_{S_1,S_2} \overline{\kappa}^! \check{u}_! \mathscr{F}.$$

By construction, the square

$$\begin{tikzcd}
\overline{\mathit{\Pi}}^{S_1,S_2}\mathbf{X} \arrow[r, "\pi"] \arrow[d, "\kappa \circ i_{S_1,S_2}"] & Z \arrow[d] \\
\overline{\Gamma} \arrow[r, "\pi_1"] & Z_1
\end{tikzcd}$$

is Cartesian. Let us call the right vertical map f. Base change yields

$$\pi_* i^!_{S_1,S_2} \overline{\kappa}^! \breve{u}_! \mathscr{F} = f^! \pi_{1,*} \breve{u}_! \mathscr{F}.$$

Finally, let $i_{Z_1} : Z_1 \hookrightarrow \overline{\Gamma}$ be the inclusion. Applying the contraction principle again we obtain

$$f^! \pi_{1,*} \breve{u}_! \mathscr{F} = f^! i^*_{Z_1} \breve{u}_! \mathscr{F} = 0. \qquad \square$$

Proof of Theorem 6.1. By Lemma 6.5 the closed subschemes $\overline{\Pi}^S_{\mathbf{Y}}\mathbf{X}$ for $S \in \operatorname{Stab}\mathbf{X}$ cover $\overline{\Pi}_{\mathbf{Y}}\mathbf{X}$. If $S_1, S_2 \in \operatorname{Stab}\mathbf{X}$, then

$$\overline{\Pi}^{S_1}_{\mathbf{Y}}\mathbf{X} \cap \overline{\Pi}^{S_2}_{\mathbf{Y}}\mathbf{X} = \overline{\Pi}^{S_1 \cap S_2,\, S_1 S_2}_{\mathbf{Y}}.$$

Thus the theorem follows from an iterated Mayer-Vietoris argument using Lemmas 6.6. $\square$

REFERENCES

[A] Pramod N. Achar. “Notes on exotic and perverse-coherent sheaves”. In: *ArXiv e-prints* (Sept. 2014). arXiv: 1409.7346 [math.RT].

[AB] Dmitry Arinkin and Roman Bezrukavnikov. “Perverse coherent sheaves”. In: *Moscow Mathematical Journal* 10.1 (2010), pp. 3–29, 271. ISSN: 1609-3321.

[AG] Dima Arinkin and Dennis Gaitsgory. “Singular support of coherent sheaves and the geometric Langlands conjecture”. In: *Selecta Mathematica. New Series* 21.1 (2015), pp. 1–199. ISSN: 1022-1824. DOI: 10.1007/s00029-014-0167-5.

[B1] David Ben-Zvi. *What is the equivariant cohomology of a group acting on itself by conjugation?* MathOverflow. Apr. 9, 2010. URL: http://mathoverflow.net/a/20849 (visited on 04/27/2015).

[B2] Roman Bezrukavnikov. “Perverse coherent sheaves (after Deligne)”. In: *ArXiv e-prints* (May 2000). arXiv: math/0005152.

[B3] Roman Bezrukavnikov. “Quasi-exceptional sets and equivariant coherent sheaves on the nilpotent cone”. In: *Representation Theory. An Electronic Journal of the American Mathematical Society* 7 (2003), 1–18 (electronic). ISSN: 1088-4165. DOI: 10.1090/S1088-4165-03-00158-4.

[B4] Alexander Braverman. *Lectures on algebraic D-modules*. URL: http://www.math.harvard.edu/~gaitsgde/grad_2009/Dmod_brav.pdf.

[B5] Tom Bridgeland. “Derived categories of coherent sheaves”. In: *International Congress of Mathematicians. Vol. II*. Eur. Math. Soc., Zürich, 2006, pp. 563–582.

[BBD] Alexander A. Beilinson, Joseph Bernstein, and Pierre Deligne. “Faisceaux pervers”. In: *Analysis and topology on singular spaces, I (Luminy, 1981)*. Astérisque 100. Paris: Soc. Math. France, 1982, pp. 5–171.

[BD] Alexander Beilinson and Vladimir Drinfeld. “Quantization of Hitchin’s integrable system and Hecke eigensheaves”. URL: http://www.math.uchicago.edu/~mitya/langlands/hitchin/BD-hitchin.pdf.

[BG] Alexander Braverman and Dennis Gaitsgory. “Geometric Eisenstein series”. In: *Inventiones Mathematicae* 150.2 (2002), pp. 287–384. ISSN: 0020-9910. DOI: 10.1007/s00222-002-0237-8.

[BIK] Dave Benson, Srikanth B. Iyengar, and Henning Krause. "Local cohomology and support for triangulated categories". In: *Annales Scientifiques de l'École Normale Supérieure*. Quatrième Série 41.4 (2008), pp. 573–619. ISSN: 0012-9593.

[BM] Roman Bezrukavnikov and Ivan Mirković. "Representations of semisimple Lie algebras in prime characteristic and the noncommutative Springer resolution". In: *Annals of Mathematics. Second Series* 178.3 (2013), pp. 835–919. ISSN: 0003-486X. DOI: 10.4007/annals.2013.178.3.2.

[BN] David Ben-Zvi and David Nadler. "Nonlinear Traces". In: *ArXiv e-prints* (May 2013). arXiv: 1305.7175 [math.AG].

[BS] M. P. Brodmann and R. Y. Sharp. *Local cohomology: an algebraic introduction with geometric applications*. Cambridge Studies in Advanced Mathematics 60. Cambridge: Cambridge University Press, 1998. ISBN: 0-521-37286-0. DOI: 10.1017/CBO9780511629204.

[C] Lee Cohn. "Differential Graded Categories are k-linear Stable Infinity Categories". In: *ArXiv e-prints* (Aug. 2013). arXiv: 1308.2587 [math.AT].

[DG1] Vladimir Drinfeld and Dennis Gaitsgory. "On some finiteness questions for algebraic stacks". In: *Geometric and Functional Analysis* 23.1 (2013), pp. 149–294. ISSN: 1016-443X. DOI: 10.1007/s00039-012-0204-5.

[DG2] Vladimir Drinfeld and Dennis Gaitsgory. "On a theorem of Braden". In: *Transformation Groups* 19.2 (2014), pp. 313–358. ISSN: 1083-4362. DOI: 10.1007/s00031-014-9267-8.

[F] Giovanni Faonte. "Simplicial nerve of an A-infinity category". In: *ArXiv e-prints* (Dec. 2013). arXiv: 1312.2127 [math.AT].

[FG] John Francis and Dennis Gaitsgory. "Chiral Koszul duality". In: *Selecta Mathematica. New Series* 18.1 (2012), pp. 27–87. ISSN: 1022-1824. DOI: 10.1007/s00029-011-0065-z.

[G] Ofer Gabber. "Notes on some t-structures". In: *Geometric aspects of Dwork theory. Vol. I, II*. Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 711–734.

[GR1] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry*. Preliminary version. URL: http://www.math.harvard.edu/~gaitsgde/GL/.

[GR2] Dennis Gaitsgory and Nick Rozenblyum. "Crystals and D-modules". In: *Pure and Applied Mathematics Quarterly* 10.1 (2014), pp. 57–154. ISSN: 1558-8599. DOI: 10.4310/PAMQ.2014.v10.n1.a2.

[H] Robin Hartshorne. *Residues and Duality*. Lecture Notes in Mathematics 20. Berlin: Springer-Verlag, 1966. vii+423.

[J] Xin Jin. "Holomorphic Lagrangian branes correspond to perverse sheaves". In: *ArXiv e-prints* (Nov. 2013). arXiv: 1311.3756 [math.SG].

[K1] Masaki Kashiwara. "t-structures on the derived categories of holonomic $\mathscr{D}$-modules and coherent $\mathscr{O}$-modules". In: *Moscow Mathematical Journal* 4.4 (2004), pp. 847–868. URL: http://mi.mathnet.ru/eng/mmj173.

[K2] Bernhard Keller. "On differential graded categories". In: *International Congress of Mathematicians. Vol. II*. Eur. Math. Soc., Zürich, 2006, pp. 151–190.

[K3] Clemens Koppensteiner. "Exact functors on perverse coherent sheaves". In: *Compositio Mathematica* FirstView (May 2015). ISSN: 1570-5846. DOI: 10.1112/S0010437X15007265.

[KS] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds*. Grundlehren der Mathematischen Wissenschaften 292. With a chapter in French by Christian Houzel, Corrected reprint of the 1990 original. Berlin: Springer-Verlag, 1994. x+512. ISBN: 3-540-51861-4.

[L1] Jacob Lurie. *Higher topos theory*. Annals of Mathematics Studies 170. Princeton, NJ: Princeton University Press, 2009. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9.

[L2] Jacob Lurie. *Higher Algebra*. Sept. 14, 2014. URL: http://www.math.harvard.edu/~lurie/.

[SGA2] Alexander Grothendieck and Michèle Raynaud. *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*. Ed. by Yves Laszlo. Documents Mathématiques (Paris) 4. Séminaire de Géométrie Algébrique du Bois Marie, 1962. Augmenté d'un exposé de Michèle Raynaud. Revised reprint of the 1968 French original. Paris: Société Mathématique de France, 2005. x+208. ISBN: 2-85629-169-4. arXiv: math/0511279.