

**Midterm 1 — February 8, 2017 Duration: 50 minutes**

*This test has 4 questions on 6 pages, for a total of 25 points.*

- Read all the questions carefully before starting to work.
- Except for Question 3a, all questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space. If you continue a problem on a different page, indicate this clearly at the bottom of the problem's original page.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Solutions

First Name: \_\_\_\_\_ Last Name: \_\_\_\_\_

Student-No: \_\_\_\_\_ Section: **201** \_\_\_\_\_

Signature: \_\_\_\_\_

Question:	1	2	3	4	Total
Points:	10	5	8	2	25
Score:					

**Student Conduct during Examinations**

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
  - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
  - (iii) purposely viewing the written papers of other examination candidates;
  - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1. Consider the curve with the parametrization

$$\vec{r}(t) = \langle e^t \sin(t), 2e^t, e^t \cos(t) \rangle.$$

2 marks

(a) If  $\vec{r}(t)$  is the position of a particle at time  $t$ , compute the velocity and speed of the particle at time  $t$ . Simplify your answer.

$$v(t) = r'(t) = \langle e^t(\sin(t) + \cos(t)), 2e^t, e^t(\cos(t) - \sin(t)) \rangle \quad \text{— velocity}$$

$$|v(t)| = \sqrt{e^{2t}(\sin^2(t) + \cos^2(t)) + 4e^{2t} + e^{2t}(\cos(t) - \sin(t))^2}$$

$$= \sqrt{e^{2t}(2\sin^2(t) + 2\cos^2(t) + 4)} = e^t \sqrt{6} \quad \text{— speed}$$

2 marks

(b) Compute the curvature  $\kappa(0)$  of  $\vec{r}$  at  $t = 0$ .

$$r''(t) = \langle e^t(\sin(t) + \cos(t) + \cos(t) - \sin(t)), 2e^t, e^t(\cos(t) - \sin(t) - \sin(t) - \cos(t)) \rangle$$

$$= \langle 2e^t \cos(t), 2e^t, -2e^t \sin(t) \rangle$$

$$r'(0) = \langle 1, 2, 1 \rangle, \quad |r'(0)| = \sqrt{6}$$

$$r''(0) = \langle 2, 2, 0 \rangle$$

$$|r'(0) \times r''(0)| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 2 & 2 & 0 \end{vmatrix} = | \langle -2, 2, 2 - 4 \rangle | = \sqrt{2^2 + 2^2 + 2^2} = 2\sqrt{3}$$

$$\kappa(0) = \frac{|r'(0) \times r''(0)|}{|r'(0)|^3} = \frac{2\sqrt{3}}{6\sqrt{6}} = \frac{2}{6\sqrt{2}} = \frac{\sqrt{2}}{6}$$

Consider the curve with the parametrization

$$\vec{r}(t) = \langle e^t \sin(t), 2e^t, e^t \cos(t) \rangle.$$

2 marks

- (c) Compute the magnitude of the normal and tangential components of the acceleration  $\vec{r}''(0)$  at  $t = 0$ .

$$\text{tangential component} = \frac{d|v|}{dt} = e^t \sqrt{6} \quad \Rightarrow \text{at } t=0: \underline{\underline{\sqrt{6}}}$$

$$\text{normal component at } t=0: \kappa(0) \cdot v(0)^2 = \frac{\sqrt{2}}{6} \cdot (\sqrt{6})^2 = \underline{\underline{\sqrt{2}}}$$

2 marks

- (d) Compute  $s(t)$ , the arclength of the curve from  $\vec{r}(0)$  to  $\vec{r}(t)$ .

$$s(t) = \int_0^t |r'(u)| du = \int_0^t \sqrt{6} e^u du = \sqrt{6} e^u \Big|_0^t = \underline{\underline{\sqrt{6} e^t - \sqrt{6}}}$$

2 marks

- (e) Reparametrize  $\vec{r}(t)$  with respect to arclength  $s$  starting at the point  $\vec{r}(0) = \langle 0, 2, 1 \rangle$ .

$$s = \sqrt{6}(e^t - 1) \quad \Rightarrow \quad \frac{s}{\sqrt{6}} + 1 = e^t - 1 \quad \Rightarrow \quad e^t = \frac{s}{\sqrt{6}} + 1 \quad \Rightarrow \quad t = \ln\left(\frac{s}{\sqrt{6}} + 1\right)$$

$$\text{so: } \vec{r}(s) = \left\langle \left(\frac{s}{\sqrt{6}} + 1\right) \cdot \sin\left(\ln\left(\frac{s}{\sqrt{6}} + 1\right)\right), 2\left(\frac{s}{\sqrt{6}} + 1\right), \left(\frac{s}{\sqrt{6}} + 1\right) \cos\left(\ln\left(\frac{s}{\sqrt{6}} + 1\right)\right) \right\rangle$$

5 marks

2. Let  $C$  be the line segment from the origin to the point  $(2, 1)$  in the plane. Compute the integral

$$\int_C xy^2 ds.$$

parametrization of  $C$ :

$$\vec{r}(t) = \langle 2t, t \rangle, \quad 0 \leq t \leq 1, \quad \vec{r}'(t) = \langle 2, 1 \rangle, \quad |\vec{r}'(t)| = \sqrt{5}$$

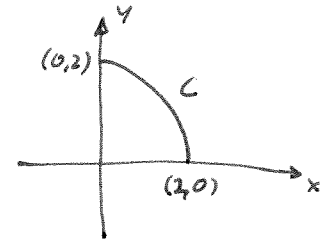
$$\text{so } \int_C xy^2 ds = \int_0^1 2t \cdot t^2 \cdot \sqrt{5} dt = \int_0^1 2\sqrt{5} t^3 dt = \left. \frac{\sqrt{5}}{2} t^4 \right|_{t=0}^1 = \underline{\underline{\frac{\sqrt{5}}{2}}}$$

3 marks

3. (a) Which of the following is a parametrization of the quarter-circle with radius two in the upper right quadrant ( $x \geq 0, y \geq 0$ ).

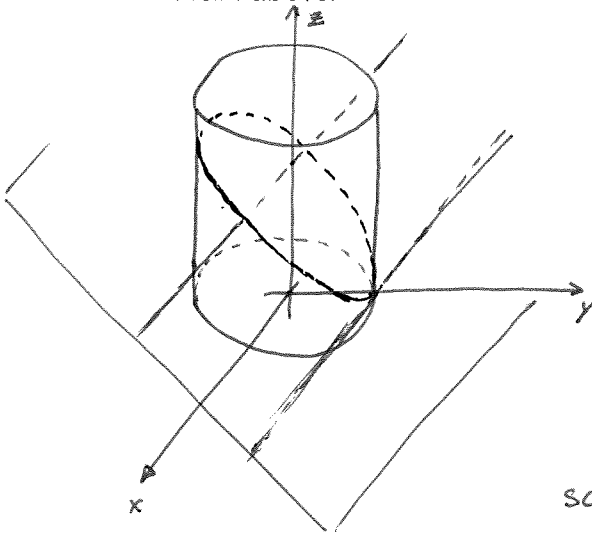
You get +1/2 mark for each correct line, -1/2 mark for each incorrect line and 0 marks for each empty line.

	Yes	No
$\vec{r}(t) = \langle 2 \cos(2t), 2 \sin(2t) \rangle, 0 \leq t \leq \pi/4$	X	
$\vec{r}(t) = \langle -2 \sin(t), 2 \cos(t) \rangle, 0 \leq t \leq \pi/2$		X
$\vec{r}(t) = \langle 2t, 2\sqrt{1-t^2} \rangle, 0 \leq t \leq 1$	X	
$\vec{r}(t) = \langle 2\sqrt{1-t^2}, 2t \rangle, 0 \leq t \leq 1$	X	
$\vec{r}(t) = \langle 2 \sin(t), 2 \cos(t) \rangle, 0 \leq t \leq \pi/2$	X	
$\vec{r}(t) = \langle 2t, \sqrt{2-t^2} \rangle, 0 \leq t \leq 1$		X



5 marks

- (b) The curve  $C$  is the intersection of the cylinder  $x^2 + y^2 = 4$  and the plane  $2y + z = 4$ . Give a parametrization  $\vec{r}(t)$  of  $C$  such that  $C$  is oriented counter-clockwise when viewed from above.



projection of  $C$  onto  $xy$ -plane is  
a circle  $\rightarrow x = 2 \cos(t)$   
 $y = 2 \sin(t)$   $0 \leq t \leq 2\pi$  counter-clockwise  
gives counter-clockwise  
orientation

$$z = 4 - 2y = 4 - 2 \sin(t)$$

so  $\vec{r}(t) = \langle 2 \cos(t), 2 \sin(t), 4 - 2 \sin(t) \rangle$  is  
the desired orientation.

2 marks 4. Suppose  $\vec{r}(t)$  is a parametrized curve such that

$$\frac{d}{dt}(\vec{r}(t) \times \vec{r}'(t)) = 0. \quad \text{and } \vec{r}(t) \neq \langle 0, 0 \rangle.$$

Show that there exists a function  $f(t)$  such that

$$\vec{a}(t) = \vec{r}''(t) = f(t)\vec{r}(t).$$

$$\frac{d}{dt}(\vec{r}(t) \times \vec{r}'(t)) = \underbrace{\vec{r}'(t) \times \vec{r}'(t)}_{=0} + \vec{r}(t) \times \vec{r}''(t) \stackrel{!}{=} 0$$

$$\text{so } \vec{r}(t) \times \vec{r}''(t) = 0, \quad \cancel{\neq}$$

so  $\vec{r}(t)$  is parallel to  $\vec{r}''(t)$

$$\text{set } f(t) = \frac{|\vec{r}''(t)|}{|\vec{r}(t)|}, \quad \text{then } \vec{r}''(t) = f(t) \cdot \vec{r}(t).$$