NOTES FOR MATH 532 – ALGEBRAIC GEOMETRY I

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0. INTRODUCTION

1. AFFINE ALGEBRAIC SETS

Throughout this section k will be an arbitrary field.

1.1. ZERO LOCI OF POLYNOMIALS

Definition 1.1. *Affine n-space* of k is the set of all n-tuples of elements of k:

$$\mathbb{A}^n = \mathbb{A}_k^n = \{(a_1, \dots, a_n) \in k^n\}.$$

Given any collection of polynomials $S \subseteq k[x_1, \dots, x_n]$ their *zero set* is

$$Z(S) = \{(a_1, \dots, a_n) \in \mathbb{A}^n : f(a_1, \dots, a_n) = 0 \text{ for all } f \in S\}.$$

Subset of $\mathbb{A}^n$ which are of this form are called *algebraic subsets* of $\mathbb{A}^n$. If $S = \{f_1, \dots, f_r\}$ is a finite set, then we write $Z(f_1, \dots, f_r) = Z(S)$.

Example 1.2. Some example of algebraic subsets of $\mathbb{A}^n$ are:

(i) $\mathbb{A}^n = Z(0)$,

(ii) $\emptyset = Z(1)$,

(iii) any point $\{(a_1, \dots, a_n)\} = Z(x - a_1, \dots, x - a_n)$.

(iv) any linear subspace of $\mathbb{A}^n$. ○

Remark 1.3. The zero set of a polynomial depends on the base field. For example, $Z(x^2 + 1) = \emptyset$ in $\mathbb{A}^1_{\mathbb{R}}$, but consists of two points in $\mathbb{A}^1_{\mathbb{C}}$ and of one point in $\mathbb{A}^1_{\mathbb{F}_2}$.

If f and g vanish on a subset X of $\mathbb{A}^n$, then so do $f + g$ and $h \cdot f$ for any $h \in k[x_1, \dots, x_n]$. Thus the zero set $Z(S)$ only depends on the ideal generated by S. The Hilbert basis theorem implies that $k[x_1, \dots, x_n]$ is Noetherian, i.e. every ideal of $k[x_1, \dots, x_n]$ can be generated by finitely many elements. In particular, every algebraic set is defined by finitely many polynomials.

Definition 1.4. For any subset $X \subset \mathbb{A}^n$ we call

$$I(X) = \{f \in k[x_1, \dots, x_n] : f(P) = 0 \text{ for all } P \in X\}$$

the *ideal of* X.

By the above discussion above, $I(X)$ is indeed an ideal of $k[x_1, \dots, x_n]$.

Lemma 1.5. *If $S_1 \subseteq S_2 \subseteq k[x_1, \dots, x_n]$, then $Z(S_2) \subseteq Z(S_1) \subseteq \mathbb{A}^n$. Conversely, if $X_1 \subseteq X_2 \subseteq \mathbb{A}^n$, then $I(X_2) \subseteq I(X_1) \subseteq k[x_1, \dots, x_n]$.*

Example 1.6. By this lemma, points in $\mathbb{A}^n$ should correspond to maximal ideals. Thus we expect that the maximal ideals of $k[x_1, \dots, x_n]$ are all of the form $(x_1 - a_1, \dots, x_n - a_n)$. Clearly this cannot be true if k is not algebraically closed. On the other hand, the Nullstellensatz will show that if k is algebraically closed then all maximal ideals are of this form. ○

Lemma 1.7.

(i) If $\{S_i\}$ is a family of subsets of $k[x_1, \dots, x_n]$, then $\bigcap_i Z(S_i) = Z(\bigcup S_i)$.

(ii) If S_1, $S_2 \subseteq k[x_1, \dots, x_n]$, then $Z(S_1) \cup Z(S_2) = Z(S_1 S_2)$.

In particular, arbitrary intersections and finite unions of algebraic subsets of $\mathbb{A}^n$ are algebraic subsets.

Proof. The first statement is obvious.

For the second statement let us first prove the inclusion "$\subseteq$". If $x \in Z(S_1) \cup Z(S_2)$, then $x \in Z(S_1)$ or $x \in Z(S_2)$. So for any $f_1 \in S_1$ and $f_2 \in S_2$ we have $f_1(x) = 0$ or $f_2(x) = 0$. Thus $f_1 f_2(x) = 0$.

Conversely let $x \notin Z(S_1) \cup Z(S_2)$, then there are $f_1 \in S_1$ and $f_2 \in S_2$ with $f_1(x) \neq 0$ and $f_2(x) \neq 0$. Hence also $f_1 f_2(x) \neq 0$ and $x \notin Z(S_1 S_2)$. □

Exercise 1.8. Show that if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are ideals in $k[x_1, \dots, x_n]$ then

$$V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2) = V(\mathfrak{a}_1 \mathfrak{a}_2) = V(\mathfrak{a}_1 \cap \mathfrak{a}_2).$$

Since further $\emptyset$ and $\mathbb{A}^n$ are algebraic subsets, the set of all algebraic subsets of $\mathbb{A}^n$ satisfies the axioms for the closed subsets of a topology on $\mathbb{A}^n$.

Definition 1.9. The *Zariski topology* on $\mathbb{A}^n$ is the topology on $\mathbb{A}^n$ whose closed subsets are exactly the algebraic subsets.

Remark 1.10. The Zariski topology looks very different from the Euclidean topology we are used to. For example, proper closed subsets satisfy at least one polynomial equation and are therefore at least one dimension smaller than the whole space. In particular, nontrival closed subsets of $\mathbb{A}^1$ are exactly the finite sets. Conversely, any two non-empty open subsets of $\mathbb{A}^n$ have non-empty intersection.

Lemma 1.11. *For any $Y \subseteq \mathbb{A}^n$ we have $Z(I(Y)) = \overline{Y}$.*

Proof. Clearly $Y \subseteq Z(I(Y))$, so also $\overline{Y} \subseteq Z(I(Y))$. So we only have to show that $Z(I(Y))$ is contained in any closed subset that contains Y. Let $W \supseteq Y$ be closed, say $W = Z(\mathfrak{a})$. Then $Z(\mathfrak{a}) \supseteq Y$ and hence $I(Z(\mathfrak{a})) \subseteq I(Y)$. By definition, $\mathfrak{a} \subseteq I(Z(\mathfrak{a}))$, so also $\mathfrak{a} \subseteq I(Y)$. Hence $W = Z(\mathfrak{a}) \supseteq Z(I(Y))$. □

Example 1.12. Let $f \in k[x_1, \dots, x_n]$ and consider the *open* subset $U = \mathbb{A}^n - Z(f)$ of $\mathbb{A}^n$. We claim that U is an affine algebraic set. Indeed we have a bijection of U with the closed subset $Z(1 - f x_{n+1})$ of $\mathbb{A}^{n+1}$ given by

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, \frac{1}{f(x_1, \dots, x_n)})$$

with inverse "forget x_{n+1}". One checks that the induced topologies on U and $Z(1 - f x_{n+1})$ are mapped to each other (the closed subset $U \cap Z(\mathfrak{a})$ maps to $Z(\mathfrak{a}, 1 - f x_{n+1})$), so that this bijection is actually a homeomorphism. Thus we see that the property of being closed in affine space is not invariant under the choice of embedding. ○

1.2. THE NULLSTELLENSATZ

Before we further investigate the Zariski topology, we need to investigate the exact relationship between ideals of $k[x_1, \dots, x_n]$ and algebraic sets. We note that the operations I and Z are not mutually inverse, even when restricted to closed algebraic subsets and ideals. For example, if $k = \mathbb{R}$, then $I(Z(x^2 + 1)) = I(\emptyset) = (1) = \mathbb{R}[x]$, but over $\mathbb{C}$, $I(Z(x^2 + 1)) = (x^2 + 1)$. So, to get a satisfactory theory, we will have to restrict to algebraically closed fields.

But even then, the I is in general not the inverse operation to Z. For example $I(Z(x^2)) = (x)$. However, the Nullstellensatz will tell us that this kind of problem with exponents is essentially the only difficulty.

First, recall that the radical of an ideal $\mathfrak{a}$ in a commutative ring R is

$$\sqrt{\mathfrak{a}} = \{f \in R : f^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\}.$$

Further, an ideal $\mathfrak{a}$ is called *radical* if $\sqrt{\mathfrak{a}} = \mathfrak{a}$.

Theorem 1.13 (Nullstellensatz)**.** *Let k be algebraically closed and $\mathfrak{a}$ be an ideal of $k[x_1, \dots, x_n]$. Then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

We will not prove the Nullstellensatz in full generality here, but we will prove it for $k = \mathbb{C}$ in Section 1.4. For the general case see for example [E, Theorem 1.6 on p. 134] or [M, Theorem I.1]. However, before we do so let us write down some consequences of the Nullstellensatz. The name "Nullstellensatz" is German for "theorem of zeros" (or "theorem of zero loci"). The following corollary explains where this name comes from.

Corollary 1.14 (weak Nullstellensatz)**.** *Let k be algebraically closed and let $\mathfrak{a}$ be a proper ideal of $k[x_1, \dots, x_n]$. Then the polynomials in $\mathfrak{a}$ have a common zero, i.e. $Z(\mathfrak{a}) \neq \emptyset$.*

Proof. If $Z(\mathfrak{a}) = \emptyset$, then we have $1 \in I(Z(\mathfrak{a}))$. Hence, by the Nullstellensatz, we have $1 \in \sqrt{\mathfrak{a}}$. But then already $1 \in \mathfrak{a}$, which contradicts the assumption. □

Corollary 1.15. *If k is algebraically closed, then Z and I are mutually inverse, order-reversing bijections between closed subsets of $\mathbb{A}^n$ and radical ideals of $k[x_1, \dots, x_n]$.*

Proof. This is a direct consequence of the Nullstellensatz combined with lemmas 1.5 and 1.11. □

Corollary 1.16. *If k is algebraically closed, then all maximal ideals of $k[x_1, \dots, x_n]$ are of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$.*

Proof. Clearly, maximal ideals are radical, so by Corollary 1.15 they correspond to minimal closed subsets of $\mathbb{A}^n$, i.e. to points. Thus by Example 1.2(iii), they are of the stated form. □

1.3. IRREDUCIBILITY AND DIMENSION

From now on we will always assume that the field k is algebraically closed.

Example 1.17. Consider the algebraic subset of $\mathbb{A}^2$ defined by $xy = 0$, i.e. the coordinate cross. Intuitively this decomposes into a union of two closed subsets: the x-axis ($y = 0$) and the y-axis ($x = 0$). In the usual topology a statement like this doesn't make much sense: the coordinate cross decomposes in many different ways into closed subsets. On the other hand in the Zariski topology this is really the only such decomposition. ○

Definition 1.18. A non-empty topological space is called *irreducible* if it cannot be written as the union of two proper closed subspaces.

Example 1.19. So in the Zariski topology the coordinate cross $xy = 0$ is not irreducible, but the affine line $\mathbb{A}^1$ is (since all proper closed subsets are finite collection of points). ○

An irreducible affine algebraic set is sometimes called an *affine variety*, but be aware that the word "variety" is used with a different meaning by some authors (e.g. it might mean *any* algebraic set). For this reason we will avoid using it in this course.

Lemma 1.20. *An algebraic subset X of $\mathbb{A}^n$ is irreducible if and only if its ideal $I(X)$ is a prime ideal.*

Proof. First assume that X is irreducible. If $fg \in I(X)$, then $X \subseteq Z(fg) = Z(f) \cup Z(g)$. Hence $X = (X \cap Z(f)) \cup (X \cap Z(g))$. As X is irreducible, either $X = X \cap Z(f)$ or $X \cap Z(g)$. Thus either $X \subseteq Z(f)$ or $X \subseteq Z(g)$ and hence $f \in I(X)$ or $g \in I(X)$. This shows that $I(X)$ is a prime ideal.

Conversely assume that $I(X) = \mathfrak{p}$ is a prime ideal and suppose that $X = X_1 \cup X_2$. Then $I(X) = I(X_1) \cap I(X_2)$. But a prime ideal cannot be written as a non-trivial intersection of two other ideals. So either $I(X) = I(X_1)$ or $I(X) = I(X_2)$ and hence $X = X_1$ or $X = X_2$. □

Definition 1.21. A topological space X is called *Noetherian* if it satisfies the *descending chain condition* for closed subsets: Every sequence $X_1 \supseteq X_2 \supseteq \cdots$ of closed subsets of X eventually stabilizes, i.e. $Y_i = Y_{i+1}$ for all sufficiently large i.

Lemma 1.22. *$\mathbb{A}^n$ is a Noetherian topological space. Hence every affine algebraic set is a Noetherian topological space.*

Proof. Follows from the Nullstellensatz and the fact that $k[x_1, \ldots, x_n]$ is Noetherian (i.e. it satisfies the ascending chain condition for ideals). Every subset of a Noetherian topological space is a Noetherian topological space with the induced topology. □

Lemma 1.23. *Every Noetherian topological space X can be written as a finite union $X = X_1 \cup \cdots \cup X_r$ of irreducible closed subsets X_i. If we require that $X_i \not\subseteq X_j$ for $j \neq i$, then the closed subsets X_i are unique up to permutation.*

Proof. Assume that X cannot be written as such a union. Then in particular X is not irreducible so we can write X as the union of two proper closed subsets, $X = X_1 \cup X_1'$. Moreover the statement of the Lemma must be false for at least one of these two subset, say X_1. Write $X_1 = X_2 \cup X_2'$ and repeat the argument. We obtain an infinite sequence $X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$, a contradiction to the assumption that X is Noetherian.

To show uniqueness, assume that $X = X_1' \cup \cdots \cup X_s'$ is another such representation. Then $X_1' \subseteq X = X_1 \cup \cdots \cup X_r$, i.e. $X_1' = \bigcup (X_1' \cap X_i)$. But X_1' is irreducible, so $X_1' = X_1' \cap X_i$ for some i, say $i = 1$. Thus $X_1' \subseteq X_1$. Similarly $X_1 \subseteq X_j'$ for some j. But then $X_1' \subseteq X_1 \subseteq X_j'$, so by assumption we have $j = 1$ and $X_1 = X_1'$. Now inductively repeat the argument for $X_2 \cup \cdots \cup X_r = X_2' \cup \cdots \cup X_s'$. □

Corollary 1.24. *Every affine algebraic set X can be written as a finite union $X = X_1 \cup \cdots \cup X_r$ of irreducible closed subsets X_i. If we require that $X_i \not\subseteq X_j$ for $j \neq i$ then the X_i are unique up to permutation. The affine algebraic sets X_i are called* irreducible components *of X.*

Remark 1.25. We can obtain the same statement via the Nullstellensatz from the following fact from commutative algebra: If $\mathfrak{a} \subseteq R$ is an ideal in a Noetherian ring, then there is (up to permutation) a unique way to write $\sqrt{\mathfrak{a}}$ as the intersection of finitely many prime ideals,

$$\sqrt{\mathfrak{a}} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k,$$

such that $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for $i \neq j$. Note however that in general it is quite hard to explicitly determine the $\mathfrak{p}_i$ (or the irreducible components X_i).

Definition 1.26. Let X be a non-empty irreducible topological space. Then the *dimension* of X is the length of the longest chain $\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X$ of irreducible closed subsets of X. More generally, the dimension of a Noetherian topological space X is the supremum of the dimensions of all irreducible components of X, i.e.

$$\dim X = \sup\{n : \text{there exists a chain } \emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \text{ of closed subsets of } X\}.$$

Example 1.27. The dimension of $\mathbb{A}^1$ is 1, since single points are the only non-empty proper closed irreducible subsets of $\mathbb{A}^1$. ○

Recall that the *Krull dimension* of a commutative ring is supremum of the lengths of all chains of prime ideals in the ring.

Proposition 1.28. *The dimension of a closed subset X of $\mathbb{A}^n$ equals the Krull dimension of the ring $k[x_1, \dots, x_n]/I(X)$.*

Proof. The dimension of X is the supremum of the lengths of the all chains of irreducible closed subsets of X. By the Nullstellensatz and Lemma 1.20, this is equal to the supremum of the lengths of all chains of prime ideals $I(X) \subsetneq p_1 \subsetneq p_2 \subsetneq \cdots \subsetneq k[x_1, \dots, x_n]$. But this is nothing but the Krull dimension of $k[x_1, \dots, x_n]/I(X)$. □

Remark 1.29. This is our first hint that the algebra $k[x_1, \dots, x_n]/I(X)$ and its prime spectrum plays an important role in understanding the geometry of X.

Example 1.30. It is a (non-trivial!) fact from commutative algebra that the Krull-dimension of $k[x_1, \dots, x_n]$ is n. Thus the dimension of $\mathbb{A}^n$ is n, as expected. ○

1.4. PROOF OF THE NULLSTELLENSATZ FOR $k = \mathbb{C}$

Recall the following definition from field theory:

Definition 1.31. Let L/K be a field extension and let $B \subseteq L$ be a set of elements of L. Then B is *algebraically independent* over K if for every integer n, every non-zero polynomial $f \in K[x_1, \dots, x_n]$ and any set $b_1, \dots, b_n$ of distinct elements of B we have $f(b_1, \dots, b_n) \neq 0$. An algebraically independent set $B \subseteq L$ is called a *transcendence basis* of L over K if it is algebraically independent and L is algebraic over $K(B)$. All transcendence bases of L/K have the same cardinality. The *transcendence degree* of L/K is the cardinality of any transcendence basis.

Alternatively a set $B \subseteq L$ is algebraically independent if there exists a field homomorphism (which is automatically a monomorphism) $K(\{x_b\}_{b \in B}) \to L$ sending x_b to b. The transcendence degree of $\mathbb{C}$ over $\mathbb{Q}$ is the cardinality of the continuum (and in particular infinite). This follows from the fact that $\overline{\mathbb{Q}}$ is countable.

Proof of the weak Nullstellensatz for $\mathbb{C}$. It clearly suffices to prove the statement for $\mathfrak{a}$ a maximal ideal.

Since $\mathbb{C}[x_1, \dots, x_n]$ is Noetherian, we can write $\mathfrak{a} = (f_1, \dots, f_k)$. Let K be the subfield of $\mathbb{C}$ obtained by adjoining to $\mathbb{Q}$ all the coefficients of the f_i. Let $\mathfrak{a}_0 = \mathfrak{a} \cap K[x_1, \dots, x_n]$. We note that all f_i are contained in $\mathfrak{a}_0$, hence $\mathfrak{a} = \mathfrak{a}_0 \cdot \mathbb{C}[x_1, \dots, x_n]$. Further, $\mathfrak{a}_0$ is maximal in $K[x_1, \dots, x_n]$: indeed if $\mathfrak{a}_0 \subsetneq \mathfrak{a}_0' \subsetneq K[x_1, \dots, x_n]$ then $\mathfrak{a} \subseteq \mathfrak{a}_0'\mathbb{C}[x_1, \dots, x_n] \subseteq \mathbb{C}[x_1, \dots, x_n]$, where the inclusions are proper since $(a_0'\mathbb{C}[x_0, \dots, x_n]) \cap K[x_0, \dots, x_n] = a_0'$.

The field $K[x_1, \dots, x_n]/\mathfrak{a}_0$ is of finite transcendence degree over K, which in turn is of finite transcendence degree over $\mathbb{Q}$. Thus there exists an embedding

$$\varphi\colon K[x_1, \dots, x_n] \hookrightarrow \mathbb{C}$$

fixing K. Let $a_i = \varphi(x_i)$. Then

$$f_i(a_1, \dots, a_n) = f_i(\varphi(x_1), \dots, \varphi(x_n)) = \varphi(f_i(x_1, \dots, x_n)) = \varphi(0) = 0, \qquad \text{for } i = 1, \dots, k,$$

i.e. $(a_1, \dots, a_n) \in Z(\mathfrak{a})$. □

Deduction of the Nullstellensatz from the weak Nullstellensatz (Rabinowitsch Trick [R]). First let $f \in \sqrt{\mathfrak{a}}$. Then f^r vanishes on $Z(\mathfrak{a})$ and hence so does f. Thus $\sqrt{\mathfrak{a}} \subseteq I(Z(\mathfrak{a}))$.

Conversely, let f be a nonzero element of $I(Z(\mathfrak{a}))$. We have to show that there exists a number r such that $f^r \in \mathfrak{a}$.

Since $\mathbb{C}[x_1, \dots, x_n]$ is Noetherian we can write $\mathfrak{a} = (f_1, \dots, f_k)$. Consider the ideal $(f_1, \dots, f_k, 1-yf)$ in $\mathbb{C}[x_1, \dots, x_n, y]$. Since f vanishes whenever $f_1, \dots, f_k$ vanish, the polynomials $f_1, \dots, f_n$, $1-yf$ do not have any common zeros. Thus by the weak Nullstellensatz $(f_1, \dots, f_k, 1-yf) = \mathbb{C}[x_1, \dots, x_n, y]$, i.e. there exist $g_0, \dots, g_1 \in \mathbb{C}[x_1, \dots, x_n, y]$ such that

$$1 = g_0(x_1, \dots, x_n, y)(1 - yf(x_1, \dots, x_n)) + \sum_{i=1}^{k} g_i(x_1, \dots, x_n, y) f_i(x_1, \dots, x_n).$$

This relationship remains true if we substitute $y = 1/f$, so that

$$1 = \sum_{i=1}^{k} g_i(x_1, \dots, x_n,\ 1/f(x_1, \dots, x_n)) f_i(x_1, \dots, x_n)$$

in $\mathbb{C}(x_1, \dots, x_n)$. Rewriting the right-hand side with a common denominator we obtain

$$1 = \frac{\sum_{i=1}^{k} h_i(x_1, \dots, x_n) f_i(x_1, \dots, x_n)}{f(x_1, \dots, x_n)^r}$$

for some $h_i \in \mathbb{C}[x_1, \dots, x_n]$ and $r \in \mathbb{N}$. Hence

$$f^r = \sum_{i=1}^{k} h_i f_i \in \mathfrak{a}.$$ □

1.5. MORPHISMS AND FUNCTIONS

In the previous sections we defined affine algebraic sets and studied some of their properties. To get a full theory we also need to define morphisms between algebraic sets, i.e. we need to define the category of affine algebraic sets.

[Intro to morphisms]

Definition 1.32. Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine algebraic sets. Then a map $f: X \to Y$ (of sets) is called a *morphism* or a *regular map* if there exist polynomials $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ such that $f(x) = (f_1(x), \dots, f_m(x))$ for all $x \in X$. Further f is an *isomorphism* if it is bijective and its inverse is again a morphism.

Examples 1.33.

- The projection map $(x, y) \mapsto x$ defines a morphism from the curve $xy = 1$ to $\mathbb{A}^1$. More generally any restriction of a projection map $\mathbb{A}^n \to \mathbb{A}^m$ is a morphism.
- The map $f(t) = (t^2, t^3)$ is a regular map of the affine line $\mathbb{A}^1$ to the curve $y^2 = x^3$.
- The projection map $\{y = x^2\} \to \mathbb{A}^1$ is an isomorphism. ○

Exercise 1.34. Every morphism is continuous with respect to the Zariski topology.

Remark 1.35. The converse is not true: not every continuous map is a morphism.

Exercise 1.36. The composition of two morphisms is again a morphism.

Definition 1.37. The *category* $\mathbf{Aff}_k$ *of affine algebraic sets over* k is the category whose objects are closed subsets $X \subseteq \mathbb{A}^n$ for any n and whose Hom-sets consist of morphisms as defined above.

Definition 1.38. A regular map $X \to \mathbb{A}^1$ is called a *regular function*. We denote the set of regular functions on X by

$$\mathcal{O}(X) = \mathrm{Hom}_{\mathbf{Aff}_k}(X, \mathbb{A}^1_k).$$

Clearly, $\mathcal{O}(X)$ has the structure of a k-algebra. It is sometimes called the *affine coordinate ring* of X and is often denoted by $k[X]$.

Lemma 1.39. *Let X be an affine algebraic subset of $\mathbb{A}^n$. Then there is an isomorphism of k-algebras $\mathcal{O}(X) \cong k[x_1, \dots, x_n]/I(X)$.*

Proof. Consider the map $k[x_1, \dots, x_n] \to \mathcal{O}(X)$, sending a polynomial to the regular function defined by it. The kernel of this (obviously surjective) map consists exactly of the polynomials vanishing on all of X, i.e. it is $I(X)$. □

From the lemma it is obvious that $\mathcal{O}(X)$ is of finite type (i.e. finitely generated as an algebra) and reduced (i.e. if $f^n = 0$, then $f = 0$).

Let now $f: X \to Y$ be a morphism and let $\alpha \in \mathcal{O}(Y)$ be a regular function on X. Then $f^*\alpha = \alpha \circ f$ is a regular function on X as

$$f^*\alpha(x_1, \dots, x_n) = \alpha(f_1(x_1, \dots, x_n), \ \dots, f_m(x_1, \dots, x_n))$$

is again a polynomial function. It is now easy to check the following lemma:

Lemma 1.40. *If $f: X \to Y$ is a morphism of affine algebraic sets, then $f^*: \mathcal{O}(Y) \to \mathcal{O}(X)$ is a homomorphism of k-algebras. If further $g: Y \to Z$ is another morphism, then $(g \circ f)^* = f^* \circ g^*$. In other words, there is a contravariant functor*

$$\mathbf{Aff}_k \to \mathbf{Alg}_k, \quad X \mapsto \mathcal{O}(X), \quad f \mapsto f^*.$$

We will denote this functor simply by $\mathcal{O}: \mathbf{Aff}_k \to \mathbf{Alg}_k$.

Example 1.41. Example about bijective morphism ≠ isomorphism [G1, Example 2.3.8]. ○

Lemma 1.42. *The functor $\mathcal{O}$ is fully faithful, i.e. for any affine algebraic sets X and Y it induces a bijection*

$$\mathrm{Hom}_{\mathbf{Aff}_k}(X, Y) \cong \mathrm{Hom}_{\mathbf{Alg}_k}(\mathcal{O}(X), \mathcal{O}(Y)).$$

Proof. We need to show that given any homomorphism $\varphi: \mathcal{O}(Y) \to \mathcal{O}(X)$, there is a unique morphism $f: X \to Y$ such that $\varphi = f^*$.

Let us write $\mathcal{O}(X) = k[x_1, \dots, x_n]/\mathfrak{a}$ and $\mathcal{O}(Y) = k[y_1, \dots, y_m]/\mathfrak{b}$. Then $\varphi: \mathcal{O}(Y) \to \mathcal{O}(X)$ is determined by the images of the y_i, say

$$\varphi(y_i) = f_i(x_1, \dots, x_n) + \mathfrak{a}, \qquad f_i \in k[x_1, \dots, x_n], \qquad i = 1, \dots, m.$$

These f_i determine a morphism

$$\tilde{\varphi}': \mathbb{A}^n \to \mathbb{A}^m, \quad (x_1, \dots, x_n) \mapsto (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)).$$

We claim that $\tilde{\varphi}'$ restricts to a map $\tilde{\varphi}: Z(\mathfrak{b}) \to Z(\mathfrak{a})$. Indeed, if $f \in \mathfrak{a}$ and $\mathsf{y} \in Z(\mathfrak{b})$, then $f(\tilde{\varphi}(\mathsf{y})) = \varphi(f)(\mathsf{y}) = 0$ as $\varphi(\mathfrak{a}) \subseteq \mathfrak{b}$. Further, by construction we have $\tilde{\varphi}^* = \varphi$. Thus the functor is full.

Now let $f: X \to Y$ be morphism of affine algebraic sets. Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ and write $f = (f_1, \dots, f_m)$. Assume that $f^*: \mathcal{O}(Y) \to \mathcal{O}(X)$ is 0. Then $f^*(g) = g \circ f = 0$ for all $g \in \mathcal{O}(Y)$. If $y_1, \dots, y_m$ are the coordinates on $\mathbb{A}^m$, then in particular

$$f^*(y_i) = y_i \circ f = f_i = 0.$$

Hence $f = (f_1, \dots, f_i) = 0$. Thus the functor is faithful. □

Corollary 1.43. *The functor $\mathcal{O}$ induces an equivalence of categories*

$$\mathbf{Aff}_k \cong \{\textit{reduced finite type } k\textit{-algebras}\}.$$

Proof. It only remains to show that the essential image of $\mathcal{O}$ consists of exactly the reduced finite type algebras. We already know that $\mathcal{O}(X)$ is a reduced finite type algebra for any affine algebraic set. So let A be such an algebra. Then we can write $A = k[x_1, \dots, x_n]/\mathfrak{a}$ with $\mathfrak{a}$ radical. Let $X = Z(\mathfrak{a}) \subseteq \mathbb{A}^n$. Then by the Nullstellensatz we have $I(X) = \mathfrak{a}$ and hence $\mathcal{O}(X) \cong A$. □

Corollary 1.44. *Let $f: X \to Y$ be a continuous map. Then f is a morphism if and only if f^* pulls back regular functions to regular functions.*

[Note about locally ringed spaces, definition of morphism independent of the embedding.]

2. THE CATEGORY OF QUASIPROJECTIVE ALGEBRAIC SETS

Again, k will always be an algebraically closed field of characteristic 0.

2.1. CLOSED SUBSETS OF PROJECTIVE SPACE

[Intro to projective space. $\{xy = 1\}$ intersected with lines $y = tx$. Picture about lines through origin intersecting $x_0 = 1$.]

Definition 2.1. On $\mathbb{A}^{n+1} - \{0\}$ define an equivalence relation by

$$(a_0, \dots, a_n) \sim (\lambda a_0, \dots, \lambda a_n), \quad \lambda \in k^*,$$

i.e. two points are equivalent if they are multiples of each other. Then n-dimensional *projective space* $\mathbb{P}^n$ is the quotient $(\mathbb{A}^{n+1} - \{0\})/\sim$. The point of $\mathbb{P}^n$ corresponding to the equivalence class of $(a_0, \dots, a_n)$ in $\mathbb{A}^{n+1}$ is denoted by $(a_0 : \cdots : a_n)$.

We would like to define subsets $Z(\mathfrak{a})$ in $\mathbb{P}^n$ in the same way as for affine space. However, if $f \in k[x_0, \dots, x_n]$ is any polynomial, then we cannot simply plug in a point $(a_0 : \dots : a_n)$ as $f(a_0, \dots, a_n)$ might differ from $f(\lambda a_0, \dots, \lambda a_n)$. Thus we need to restrict to polynomials where at least the zero locus in well defined for projective points.

Definition 2.2. A polynomial $f \in k[x_0, \dots, x_n]$ is *homogeneous of degree d* if

$$f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n)$$

for all $\lambda \in k^*$. We denote the set of all homogeneous polynomials of degree d by $k[x_0, \dots, x_n]_d$.

Clearly a polynomial is homogeneous of degree d if all its monomial terms have degree d. E.g. $x_0x_1 + x_1^2$ is homogeneous, but $x_0x_1 + x_1$ is not. Thus we get a decomposition

$$k[x_0, \dots, x_n] = \bigoplus_{d \geq 0} k[x_1, \dots, x_n]_d.$$

If f is a homogeneous polynomial, then

$$f(a_0, \dots, a_n) = 0 \Leftrightarrow f(\lambda a_0, \dots, \lambda a_n) = 0$$

for any $\lambda \in k^*$. Thus the zero locus of a homogeneous polynomial is a well-defined subset of $\mathbb{P}^n$.

Definition 2.3. An ideal $\mathfrak{a} \subseteq k[x_0, \dots, x_n]$ is called a *homogeneous ideal* if $\mathfrak{a} = \bigoplus_{d \geq 0} \mathfrak{a} \cap k[x_0, \dots, x_n]_d$.

Note that every homogeneous ideal can be generated by finitely many homogeneous polynomials.

Definition 2.4. Let $\mathfrak{a} \subseteq k[x_0, \dots, x_n]$ be a homogeneous ideal. The *projective zero set* of $\mathfrak{a}$ is

$$\widetilde{Z}(\mathfrak{a}) = \{(a_0 : \cdots : a_n) \in \mathbb{P}^n : f(a_0, \dots, a_n) = 0 \text{ for all homogeneous } f \in \mathfrak{a}\}.$$

If $f_1, \dots, f_r$ are homogeneous polynomial, we set $\widetilde{Z}(f_1, \dots, f_r) = \widetilde{Z}((f_1, \dots, f_r))$. Subsets of $\mathbb{P}^n$ of this form are called *(projective) algebraic sets*.

Exercise 2.5. The union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets.

Definition 2.6. We define the *Zariski topology* on $\mathbb{P}^n$ by taking the closed sets to be algebraic sets. A *quasi-projective algebraic set* is an open subset of a projective algebraic set.

Definition 2.7. Let $X \subseteq \mathbb{P}^n$ be any subset. Then the *homogeneous ideal* $\tilde{I}(X)$ is the ideal of $k[x_0, \dots, x_n]$ generated by all homogeneous polynomial f that vanish on X.

The space $\mathbb{P}^n$ can be covered by the open subsets

$$U_i = \mathbb{P}^n - Z(x_i) = \{(a_0 : \dots : a_n) : a_i \neq 0\}, \qquad i = 0, \dots, n.$$

We have a bijection of U_i with $\mathbb{A}^n = k^n$ by

$$\varphi_i : (a_0 : \cdots : a_n) \mapsto \left(\frac{a_0}{a_i}, \dots, \frac{\widehat{a_i}}{a_i}, \dots, \frac{a_n}{a_i} \right).$$

Note that this is well-defined since the ratios a_j/a_i are independent of the choice of homogeneous coordinates.

Proposition 2.8. *The map φ_i is a homeomorphism of U_i with its induced topology to $\mathbb{A}^n$.*

For non-zero $f \in k[x_1, \dots, x_n]$ of degree d we define the *homogenization* f^h of f by

$$f^h(x_0, \dots, x_n) = x_0^d f(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}) \in k[x_0, \dots, x_n].$$

Then f^h is a homogeneous polynomial. Further, if $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$ is an ideal, write $\mathfrak{a}^h$ for the ideal of $k[x_0, \dots, x_n]$ generated by all f^h for $f \in \mathfrak{a}$.

Proof. We may assume that $i = 0$. It suffices to show that φ^{-1} and φ are closed maps, i.e. map closed subsets to closed subsets.

Let $X \subset U_0$ be a closed subset and let $\overline{X}$ be its closure in $\mathbb{P}^n$. This is a closed subset, so it can be defined by finitely many homogeneous polynomial $f_1, \dots, f_r$. Let $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$ be the ideal generated by the polynomials $f_i(1, x_1, \dots, x_n)$. Then $\varphi_0(X) = Z(\mathfrak{a})$.

Conversely if $Y = Z(\mathfrak{b})$ is a closed subset of $\mathbb{A}^n$, then $\varphi_0^{-1}(Y) = U_0 \cap \widetilde{Z}(\mathfrak{b}^h)$. □

Proposition 2.9. *Let $X = Z(\mathfrak{a}) \subseteq \mathbb{A}^n$. Then the closure of $\varphi_0^{-1}(X)$ in $\mathbb{P}^n$ is $\widetilde{Z}(\mathfrak{a}^h)$.*

We usually simply write X for $\varphi_0^{-1}(X)$ and $\overline{X}$ for $\overline{\varphi_0^{-1}(X)} \subseteq \mathbb{P}^n$. We call $\overline{X}$ the *projective closure of* X.

Proof. If $P = (a_1, \ldots, a_n) = (1 : a_1 : \cdots : a_n) \in X$, then $f^h(P) = 1^d f(a_1, \ldots, a_n) = 0$ for all $f \in \mathfrak{a}$. Thus $X \subseteq \widetilde{Z}(\mathfrak{a}^h)$.

We have to show that $\widetilde{Z}(\mathfrak{a}^h)$ is the smallest closed set containing X. Thus let $Y \supseteq X$ be any closed set. We have to prove that Y contains $\widetilde{Z}(\mathfrak{a}^h)$. Write $Y = \widetilde{Z}(\mathfrak{b})$ for some homogeneous ideal $\mathfrak{b}$.

We note that any homogeneous polynomial in $x_0, \ldots, x_n$ can be written as $x_0^d f^h$ for some $f \in k[x_1, \ldots, x_n]$ and some $d \in \mathbb{N}$. Thus take any element of $\mathfrak{b}$ and write it in this form. Then $x_0^d f^h$ is zero on X, and since $x_0 \neq 0$ on $X \subseteq U_0$ this implies that already f is zero on X. Thus $f \in I(X) = \sqrt{\mathfrak{a}}$, i.e. $f^m \in \mathfrak{a}$ for some $m \in \mathbb{N}$. Since $(f_1 f_2)^h = f_1^h f_2^h$, this implies that $(f^m)^h = (f^h)^m \in \mathfrak{a}^h$ and hence also $x_0^d f^h \in \sqrt{\mathfrak{a}^h}$.

Thus $\mathfrak{b} \subseteq \sqrt{\mathfrak{a}^h}$ and hence

$$Y = \widetilde{Z}(\mathfrak{b}) \supseteq \widetilde{Z}(\sqrt{\mathfrak{a}^h}) = \widetilde{Z}(\mathfrak{a}^h). \qquad \square$$

Example 2.10. $(f)^h = (f^h)$. Discuss $xy = 1$.

In general it does not suffice to homogenize a set of generators. Example: $(x_1, x_1^2 - x_2)$. ○

Our next goal is to obtain an analogue of the Nullstellensatz for projective algebraic sets, i.e. establish a correspondence between projective algebraic sets and radical homogeneous ideals. However we note that the ideal $(x_0, \ldots, x_n)$ is homogeneous and radical, but its zero locus is the empty set. Thus we have to exclude it.

Definition 2.11. The ideal $(x_0, \ldots, x_n) \subseteq k[x_0, \ldots, x_n]$ is called the *irrelevant* ideal.

Theorem 2.12 (Projective Nullstellensatz)**.** *$\widetilde{Z}$ and $\widetilde{I}$ set up an order-reversing bijection between closed algebraic subsets of $\mathbb{P}^n$ and radical homogeneous ideals of $k[x_0, \ldots, x_n]$ except for the irrelevant ideal.*

Proof. As in the affine case, we always have $\widetilde{Z}(\widetilde{I}(X)) = X$ for any closed subset X of $\mathbb{P}^n$. So we only have to prove that if $\mathfrak{a}$ is a homogeneous radical ideal of $k[x_0, \ldots, x_n]$, then $\widetilde{I}(\widetilde{Z}(\mathfrak{a})) = \mathfrak{a}$. The inclusion $\supseteq$ is obvious, so we only have to show the other inclusion.

Consider the *affine* algebraic set $Z(\mathfrak{a})$ in $\mathbb{A}^{n+1}$ with coordinates $x_0, \ldots, x_n$. We note that $Z(\mathfrak{a})$ is invariant under the substitution

$$(x_0, \ldots, x_n) \to (\alpha x_0, \ldots, \alpha x_n)$$

for all $\alpha \in k$. Thus we have the following possible cases

(i) $Z(\mathfrak{a})$ is empty.

(ii) $Z(\mathfrak{a})$ is $\{0\}$.

(iii) $Z(\mathfrak{a})$ is a union of lines through the origin, i.e. it is the cone over the subset $\widetilde{Z}(\mathfrak{a})$ [picture].

By the affine Nullstellensatz we know that $\mathfrak{a} = I(Z(\mathfrak{a}))$. Let us analyze the three cases separately.

(i) Here we have $\mathfrak{a} = I(\emptyset) = k[x_0, \dots, x_n]$. Since $\mathfrak{a} \subseteq \tilde{I}(\widetilde{Z}(\mathfrak{a})) \subseteq k[x_0, \dots, x_n]$, we must have equality.

(ii) Here we have $\mathfrak{a} = (x_0, \dots, x_n)$, which we specifically excluded.

(iii) In this case a homogeneous polynomial vanishes on $Z(\mathfrak{a})$ if and only if it vanishes on $\widetilde{Z}(\mathfrak{a})$. Thus $\mathfrak{a} \subseteq \tilde{I}(\widetilde{Z}(\mathfrak{a}))$. □

Next we would like to define morphisms of (quasi-)projective algebraic sets. To simplify our life, we want to simply say that a morphism is a continuous function (with respect to the Zariski topology) that pulls back regular functions to regular functions. Then we only have to define when a continuous function $f : X \to \mathbb{A}^1$ is regular. Clearly we want to do it in such a way that if f is a regular function on $\mathbb{P}^n$ then its restriction $f|_{U_i}$ to the affine open subsets is a regular function on $\mathbb{A}^n$. However we run in to the following problems:

- The function $f|_{U_i}$ is polynomial, which, at least over $\mathbb{C}$, has to be bounded in order for us to be able to extend it continuously to $\mathbb{P}^n$. But the only globally bounded complex polynomials are constant functions (cf. Liouville's theorem). Thus *global* regular functions do not provide enough information to define morphisms.
- We also want to know the ring of regular functions on open subsets. But even in the affine case just restricting global polynomials (as we did for closed subsets) is not the right thing to do. Consider for example the open subset $U = \mathbb{A}^1 - \{0\} \subset \mathbb{A}^1$. As we discussed in Example 1.12, U is a closed subset of $\mathbb{A}^2$ defined by the ideal $(1 - x_0x_1)$ [picture]. Thus $\mathscr{O}(U) = k[x_1, x_2]/(1 - x_1x_2) = k[x_1, x_1^{-1}]$ is strictly bigger than $k[x_1]$.

Instead of just pulling the correct definitions out of thin air, we will first discuss a general tool to deal with this kind of situation. While not strictly speaking necessary here, we would have to talk about it eventually anyway.

2.2. SHEAVES

Definition 2.13. Let X be a topological space. A *presheaf* $\mathscr{F}$ in a category $\mathbf{C}$ is a contravariant functor

$$\mathscr{F} : \{\text{open subset of } X + \text{inclusions}\} \to \mathbf{C}.$$

Concretely, a *presheaf of rings* $\mathscr{F}$ on X consists of

- a ring $\mathscr{F}(U)$ for each open subset $U \subseteq X$,
- a ring homomorphism $\rho^V_U : \mathscr{F}(V) \to \mathscr{F}(U)$, called *restriction map* for each inclusion of open subsets $U \subseteq V$,

such that

- $\mathscr{F}(\emptyset) = 0$,
- $\rho_U^U = \mathrm{Id}_{\mathscr{F}(U)}$ for all U,
- for any inclusion $U \subseteq V \subseteq W$ of open sets we have
$$\rho_U^V \circ \rho_V^W = \rho_V^W : \mathscr{F}(W) \to \mathscr{F}(V).$$

The elements of $\mathscr{F}(U)$ are usually called *sections* of $\mathscr{F}$ over U. If $\varphi \in \mathscr{F}(V)$, then we write $\varphi|_U = \rho_U^V(\varphi)$.

A presheaf $\mathscr{F}$ is called a *sheaf*, if for any open covering $\{U_i : i \in I\}$ of an open subset U of X the diagram
$$\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$
is an equalizer. Concretely, this says that $\mathscr{F}$ has to satisfy the following gluing property: Let $U \subseteq X$ be an open set and $\{U_i : i \in I\}$ is an arbitrary open cover of U. Assume that we are given sections $\varphi_i \in \mathscr{F}(U_i)$ for all $i \in I$ that agree on overlaps (i.e. $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ for all $i, j \in I$). Then there exists a unique section $\varphi \in \mathscr{F}(U)$ such that $\varphi|_{U_i} = \varphi_i$ for all $i \in I$.

We should think of sections of sheaves as “function-like” objects with *local* conditions, i.e. conditions that can be checked on an open cover.

Example 2.14.

(i) The assignment $U \mapsto \{\text{continuous functions } U \to \mathbb{R}\}$ is a sheaf with the usual restriction maps. In the same way, if $X = \mathbb{R}^n$, we can define sheaves of differentiable functions, analytic functions, ...

(ii) On the other hand $U \mapsto \{\text{constant functions } U \to \mathbb{R}\}$ is a presheaf, but in general not a sheaf, since being constant is not a local condition. Concretely, let U_1 and U_2 be non-empty disjoint open subsets and φ_i be constant functions on U_i with different values. Then clearly φ_1 and φ_2 agree no $U_1 \cap U_2 = \emptyset$, but there is no *constant* function φ on $U_1 \cup U_2$ restricting to φ_1 and φ_2. Note however, that *locally constant* functions do form a sheaf.

(iii) If $\pi : E \to X$ is a vector bundle, we can form the sheaf of sections of E:
$$\mathscr{E}(U) = \{s : U \to E : \pi \circ s = \mathrm{Id}_U\},$$
where we usually put some condition on s (continuous, smooth, ...). It is possible to reconstruct a vector bundle from its sheaf of sections.

(iv) Let A be a ring and $x \in X$ a point. Then we can form the *skyscraper sheaf* with value A at x:
$$\mathscr{F}(U) = \begin{cases} A & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

○

Definition 2.15. Let $\mathscr{F}$ be a sheaf on X and U an open subset of X. Then the restriction $\mathscr{F}|_U$ of $\mathscr{F}$ to U is given by $\mathscr{F}|_U(V) = \mathscr{F}(V)$ for any open subset V of U.

Definition 2.16. Let x be any point of X and let $\mathscr{F}$ be a sheaf on X. Then the *stalk* of $\mathscr{F}$ at x is

$$\mathscr{F}_x = \lim_{U \ni x} \mathscr{F}(U),$$

where the limit is the direct limit over all open subsets $U \subseteq X$ containing x. Concretely, $\mathscr{F}_x$ consists of the equivalence classes of all pairs (U, φ), where U is an open subset of X and $\varphi \in \mathscr{F}(U)$, where $(U_1, \varphi_1) \sim (U_2, \varphi_2)$ if there is an open subset $V \subseteq U_1 \cap U_2$ such that $\varphi_1|_W = \varphi_2|_W$. Elements of $\mathscr{F}_x$ are called *germs* of sections of $\mathscr{F}$ at x.

We should think of germs as "local functions", i.e. functions that are defined on an arbitrary small neighborhood of x.

Example 2.17. The germ of a differentiable function at x allows us to compute the derivative of the function, as well as the value of the function at x, but not the value at any other point. On the other hand, the germ of a holomorphic functions knows about the whole Taylor expansion of the function at x, which in term determines the original function at least on some open disk containing x. ○

Definition 2.18. A *ringed space* is a topological space X together with a sheaf of rings $\mathscr{O}_X$ on it.

We interpret that ring as the ring of functions on X, and call $\mathscr{O}_X$ its *structure sheaf*.

Example 2.19. Every topological space can be made into a ringed space by considering the sheaf of continuous functions (with values in $\mathbb{R}$ or some other ring) on it. The structure sheaf of a differentiable manifold is the sheaf of smooth functions. ○

Intuitively, a morphism of ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ should be a continuous function $f: X \to Y$ that preserves the "structure", i.e. such that pulling back induces a ring homomorphism $f^*: \mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}(U))$ for any open $U \subseteq Y$. However note that unless we are dealing with actual functions with the same target on both sides, such a pullback f^* has to be an additional piece of data. Fortunately, at least until we talk about schemes, all our structure sheaves will consist of functions with value in $\mathbb{A}^1$.

2.3. AFFINE ALGEBRAIC SETS AS RINGED SPACES

We want to define a sheaf of rings on any affine algebraic set. Clearly if $X \subseteq \mathbb{A}^n$ is an affine algebraic set, then we need to set $\mathscr{O}(X) = k[x_1, \dots, x_n]/I(X)$. We also already know what the functions on the complement of the zero locus of a single polynomial should be.

Definition 2.20. Let X be an affine algebraic set and $f \in \mathscr{O}(X)$. Then we call

$$D(f) = X - Z(f) = \{x \in X : f(x) \neq 0\}$$

a *distinguished open subset* of X.

Lemma 2.21.

- *For any $f, g \in \mathscr{O}(X)$ we have $D(f) \cap D(g) = D(fg)$.*
- *Any open subset of X is the union of finitely many distinguished open subsets of X.*

In particular, the distinguished open subsets form a basis for the Zariski topology on X.

Proof.

- $f(x) \neq 0$ and $g(x) \neq 0$ is equivalent to $(fg)(x) \neq 0$.
- Let U be an open subset. Then we can write

  $$U = X - Z(f_1, \dots, f_r) = X - (Z(f_1) \cap \dots \cap Z(f_r)) = (X - Z(f_1)) \cup \dots \cup (X - Z(f_r)) = D(f_1) \cup \dots \cup D(f_r).$$

  □

Let us temporarily write $A(X) = k[x_1, \dots, x_n]/I(X)$. As we discussed earlier we want to have

$$\mathscr{O}(D(f)) = \mathscr{O}(X)[y]/(1 - fy) = \left\{ \frac{g}{f^n} : g \in A(X), n \in \mathbb{N} \right\} = A(X)_f.$$

Assume that we know the values of a presheaf $\mathscr{F}$ on a basis of the topology. Then there is a procedure, called *sheafification* that produces a sheaf which matches $\mathscr{F}$ as closely as possible (in a certain sense, expressed as a universal property). We will now go through that procedure of the structure sheaf $\mathscr{O}$ of X.

First we need to compute the stalks $\mathscr{O}_x$ of $\mathscr{O}$. Note, that in the limit defining the stalk of a sheaf we can restrict the directed set of open subsets containing a point to just the open subsets from a basis (since for every open set has a subset in the basis). Thus knowing the values of a presheaf at a basis suffices to compute all the stalks.

Lemma 2.22. *Suppose $\mathscr{F}$ is a presheaf on an affine algebraic set X such that $\mathscr{F}(D(f)) = A(X)_f$ for all $f \in A(X)$. Let $\mathfrak{m}_x = I(\{x\}) = \{f \in A(X) : f(x) = 0\}$. Then the stalk of $\mathscr{F}$ at $x \in X$ is given by the localization*

$$\mathscr{F}_x = A_{\mathfrak{m}_x} = \left\{ \frac{g}{f} : f, g \in A(X), f(x) \neq 0 \right\}.$$

Proof. Define a k-algebra homomorphism

$$\left\{ \frac{g}{f} : f, g \in A(X), f(x) \neq 0 \right\} \to \mathscr{F}_x, \quad \frac{g}{f} \mapsto (D(f), \frac{g}{f}).$$

This map is well-defined: if $\frac{g}{f} = \frac{g'}{f'}$, then there exists $h \in A(X)$ with $h(x) \neq 0$ such that $h(gf' - g'f) = 0$. But then $gf' = g'f$ on $D(h)$, so $\frac{g}{f} = \frac{g'}{f'}$ on $D(h)$ and hence $(D(f), \frac{g}{f}) \sim (D(f'), \frac{g'}{f'})$.

The map is clearly surjective. It is also injective: If $(U, \frac{g}{f})$ represents the zero of $\mathscr{F}_x$, then we by definition there is a distinguished open $x \in D(h) \subseteq U$ such that $\frac{g}{f} = 0$ on $D(h)$. But then $0 = hg = h(g \cdot 1 - 0 \cdot f)$ on all of X and hence $\frac{g}{f} = \frac{0}{1} = 0$. □

Sections of the structure sheaf of $\mathscr{O}$ should locally look like elements of the stalks. Thus we define:

Definition 2.23. Let X be an affine algebraic set. Then the *structure sheaf* $\mathscr{O} = \mathscr{O}_X$ of X is given on an open subset $U \subseteq X$ be the set of all functions $\varphi\colon U \to k$ with the following property: for every $a \in U$ there are $f, g \in A(X)$ such that for all x in some open neighborhood of a in X, $f(x) \neq 0$ and $\varphi = \frac{g(x)}{f(x)}$. The elements of $\mathscr{O}_X(U)$ are called *regular functions* on U.

Example 2.24. [G2, Example 3.5] ○

Now in general the procedure of sheafification might introduce extra sections or identify already existing sections (to make the gluing property work out correctly). Thus we have to actually show that $\mathscr{O}(D(f))$ is the expected ring.

Lemma 2.25. *Let $f \in A(X)$. Then $\mathscr{O}_X(D(f)) = A(X)_f$. In particular $\mathscr{O}_X(X) = A(X)$.*

Proof. Any fraction $\frac{g}{f^n} \in A(X)_f$ is clearly a regular function on $D(f)$, so we only need prove the inclusion $\subseteq$.

Thus let $\varphi\colon D(f) \to k$ be a regular function. For every point $a \in D(f)$ we have a local representation $\varphi = \frac{g_a}{f_a}$ for some $f_a, g_a \in A(X)$ which is valid on some open neighborhood U_a of a in $D(f)$. We can shrink the U_a to be of the form $D(h_a)$ for some $h_a \in A(X)$. We can replace the representation of $\varphi = \frac{g_a}{f_a}$ by $\frac{g_a h_a}{f_a h_a}$. Thus we can assume that g_a and f_a vanish on the $X(h_a) = D(f) - D(h_a)$. But then h_a and f_a have the same zero locus so we can further assume that $h_a = f_a$.

In summary, we can cover U by open subsets $D(f_a)$ and have $\varphi = \frac{g_a}{f_a}$ on $D(f_a)$ with $g_a = f_a = 0$ outside of $D(f_a)$.

On $D(f_a) \cap D(f_b)$ we have $\varphi = \frac{g_a}{f_a} = \frac{g_b}{f_b}$ and hence $g_a f_b = g_b f_a$. Since both sides are zero outside of $D(f_a) \cap D(g_a)$, we obtain that

$$g_a f_b = g_b f_a \text{ on } D(f) \text{ for all } a, b \in D(f).$$

The open subsets $D(f_a)$ cover $D(f)$, so there exist $a_1, \ldots, a_r$ such that $D(f) = D(f_{a_1}) \cup \cdots \cup D(f_{a_r})$ Passing to complements we obtain

$$Z(f) = \bigcap_{i=1}^{r} Z(f_{a_i}) = Z(f_{a_1}, \ldots, f_{a_r})$$

and thus

$$f \in I(V(f)) = \sqrt{(f_{a_1}, \dots, f_{a_r})}.$$

Hence we have

$$f^n = \sum_{i=1}^{r} k_i f_{a_i}$$

for some $n \in \mathbb{N}$ and $k_i \in A(X)$. Set $g = \sum_{i=1}^r k_i g_{a_i}$. We claim that $\varphi = \frac{g}{f^n}$. Indeed on $D(f_b)$ we have

$$gf_b = \sum_{i=1}^{r} k_i g_{a_i} f_b = \sum_{i=1}^{r} k_i g_b f_{a_i} = g_b f^n$$

and hence $\frac{g}{f^n} = \frac{g_b}{f_b} = \varphi$ on $D(f_b)$. □

[Summary]

Example 2.26. Let us compute the ring of regular functions on the open subset $U = \mathbb{A}^2 - \{0\} \subseteq \mathbb{A}^2$. Note that U is not a distinguished open subset. We claim that every regular function on U can be extended to a regular function on $\mathbb{A}^2$, i.e. that

$$\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 - \{0\}) = k[x_1, x_2].$$

Let $\varphi \in \mathcal{O}(U)$. We have $U = D(x_1) \cup D(x_2)$. On $D(x_1)$ we can write $\varphi = \frac{f}{x_1^m}$ and on $D(x_2)$ we can write $\varphi = \frac{g}{x_2^n}$ for some $f, g \in k[x_1, x_2]$ and $m, n \in \mathbb{N}$ such that $x_1 \nmid f$ and $x_2 \nmid g$.

Thus on $D(x_1) \cap D(x_2)$ we have $\frac{f}{x_1^m} = \frac{g}{x_2^n}$ and hence

$$f x_2^n = g x_1^m.$$

But the locus where this equation holds is the closed subset $Z(f x_2^m - g x_1^n) \supseteq D(x_1) \cap D(x_2)$, which has to be all of $\mathbb{A}^2$. Thus $f x_2^n = g x_1^m$ holds in the polynomial ring $\mathcal{O}(\mathbb{A}^2) = k[x_1, x_2]$.

Now by assumption the left side is not divisible by x_1 and hence $m = 0$. Hence $\varphi = \frac{f}{1}$ is a polynomial as claimed. ○

Definition 2.27. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces whose structure sheaves consist of k-algebras of functions to the field k. Then a *morphism* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ such that pulling back functions induces a ring homomorphism $f^* : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$ for each open subset $V \subseteq Y$.

An *algebraic set*[1] is a ringed space $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is a sheaf of functions $U \to k$, such that X has a finite open cover $X = \bigcup_{i \in I} U_i$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to an affine algebraic set with its structure sheaf. A morphism of algebraic sets is just a morphism of ringed spaces.

An irreducible algebraic set is often called a *pre-variety* (though sometimes a "pre-variety" might not be irreducible).

[1]This is not standard terminology!

Example 2.28. Any affine algebraic set is an algebraic set and morphisms of affine algebraic sets in the sense of Definition 2.27 are the same as morphisms in the sense of Definition 1.32. ○

Example 2.29. Let X be an affine algebraic set and $U \subseteq X$ an open subset. Then $(U, \mathcal{O}_X|_U)$ is an algebraic set, as it is covered by finitely many distinguished subsets of X. ○

We can extend our original description of morphisms between affine algebraic sets to open subsets.

Proposition 2.30. *Let U be an open subset of an affine variety X and let $Y \subseteq \mathbb{A}^n$ be another affine variety. Then the morphisms $f\colon U \to Y$ are exactly the maps of the form*

$$f = (\varphi_1, \dots, \varphi_n)\colon U \to Y, \quad x \mapsto (\varphi_1(x), \dots, \varphi_n(x))$$

with $\varphi_i \in \mathcal{O}_X(U)$ for all $i = 1, \dots, n$.

In particular, the morphisms $U \to \mathbb{A}^1$ are exactly the regular functions on U.

Proof. If f is any morphism and y_i the coordinate functions on $Y \subseteq \mathbb{A}^n$, then $\varphi_i = f^* y_i \in \mathcal{O}_X(f^{-1}(Y)) = \mathcal{O}_X(U)$ and $f = (\varphi_1, \dots, \varphi_n)$.

Conversely, let $f = (\varphi_1, \dots, \varphi_n)$ be of the given form. Then one checks as in the homework that f is continuous (using that if $\varphi \in \mathcal{O}_X(U)$, then $\varphi^{-1}(0)$ is closed in U [exercise]). Finally, if $\varphi \in \mathcal{O}_Y(W)$ is regular on some open subset $W \subseteq Y$, then

$$f^*\varphi\colon f^{-1}(W) \to k, \quad x \mapsto \varphi(\varphi_1(x), \dots, \varphi_n(x))$$

is again regular, since substituting quotients of polynomial functions into quotients of polynomial functions gives again quotients of polynomial functions. □

Lemma 2.31. *Let $f\colon X \to Y$ be any map between algebraic sets. Assume that there exists an open cover $\{U_i : i \in I\}$ of X such that the restrictions $f|_{U_i}\colon U_i \to Y$ are morphisms. Then f is a morphism.*

Proof. Exercise. □

2.4. PROJECTIVE SPACE AS A RINGED SPACE

For a projective algebraic set $X \subseteq \mathbb{P}^n$ let us write $S(X) = k[x_0, \dots, x_n]/\tilde{I}(X)$ for the *projective coordinate ring*. As in the affine case we would like to say that a regular function on X is given locally by a quotient of elements of $S(X)$. However, if $f, g \in S(X)$, then $\frac{g(a)}{f(a)}$ is only well-defined for any $a \in X \subseteq \mathbb{P}^n$ if the two polynomials have the same degree.

Definition 2.32. Let U be an open subset of a projective algebraic set X. A *regular function* on U is a map $\varphi\colon U \to k$ with the following property: for every $a \in U$ there exist $d \in \mathbb{N}$ and $f, g \in S(X)_d$ such that for all x in some neighborhood V of a in U, $f(x) \neq 0$ and $\varphi(x) = \frac{g(x)}{f(x)}$. The regular functions on open subsets of X form a sheaf of k-algebras $\mathcal{O}_X$.

Proposition 2.33. *Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set. Let $U_i = X - Z(x_i)$ be the standard open cover of X. Then $(U_i, \mathcal{O}_X|_{U_i})$ is an affine algebraic set for all $i = 0, \dots, n$. In particular, X is an algebraic set.*

Proof. It suffices to prove the statement for $i = 0$. Let $X = \widetilde{Z}(h_1, \dots, h_r)$ and set $g_j(x_1, \dots, x_n) = h_j(1, x_1, \dots, x_n)$ and $Y = Z(h_1, \dots, h_r)$. Recall that we have a homeomorphism

$$\varphi_0 \colon U_0 \to Y, \quad (x_0 : \dots : x_n) \mapsto (\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$$

with inverse

$$\varphi_0^{-1} \colon Y \to U_0, \quad (x_1, \dots, x_n) \mapsto (1 : x_1 : \dots : x_n).$$

A regular function on an open subset of U_0 is locally of the form $\frac{p(x_0,\dots,x_n)}{q(x_0,\dots,x_n)}$ where p and q are homogeneous polynomials of the same degree and q is nowhere vanishing. Then

$$(\varphi_0^{-1})^* \frac{p(x_0, \dots, x_n)}{q(x_0, \dots, x_n)} = \frac{p(1, x_1, \dots, x_n)}{q(1, x_1, \dots, x_n)}$$

is a quotient of to polynomials with nowhere vanishing denominator. Conversely, φ pulls back a quotient $\frac{p(x_1,\dots,x_n)}{q(x_1,\dots,x_n)}$ of two polynomials to

$$\varphi^* \frac{p(x_1, \dots, x_n)}{q(x_1, \dots, x_n)} = \frac{p(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})}{q(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})}.$$

Multiplying the numerator and denominator by $x_0^{\max(\deg p, \deg q)}$ we see that this is a fraction of two homogeneous polynomials of the same degree. $\square$

In general, in order to see whether a given function between (open subsets of) projective algebraic sets is a morphism, we have to restrict to the open subsets U_i (or some other affine cover) and check that it is a morphism there. However, there as a class of morphisms that is globally defined:

Lemma 2.34. *Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set and let $f_0, \dots, f_m \in S(X)$ be homogeneous elements of the same degree. Then on the open subset $X - \widetilde{Z}(f_0, \dots, f_m)$ these elements define a morphism*

$$f \colon U \to \mathbb{P}^m, \quad x \mapsto (f_0(x) : \dots : f_m(x)).$$

Proof. Since the f_i are homogeneous of the same degree, f is indeed well-defined: by definition of 0, the image point can never be $(0 : \dots : 0)$ and

$$(f_0(\lambda x_0 : \dots : \lambda x_n), \dots, f_m(\lambda x_0 : \dots : \lambda x_n)) = (\lambda^d f_0(x_0 : \dots : x_n) \ \dots, \lambda^d f_m(x_0 : \dots : x_n)) =$$
$$(f_0(x_0 : \dots : x_n) \ \dots, f_m(x_0 : \dots : x_n)).$$

To check that f is morphism, it suffices to check it on an open cover. So let $V_i = \{(y_0 : \cdots : y_m) \in \mathbb{P}^m : y_i = 0\}$, and $U_i = f^{-1}(V_i) = \{x \in X : f_i(x) \neq 0\}$. Then the U_i cover U and in the affine coordinates on V_i, the map $f|_{U_i}$ is given by

$$f|_{U_i} = \left(\frac{f_1}{f_i}, \dots \frac{\widehat{f_i}}{f_i}, \dots \frac{f_m}{f_i}\right).$$

Each of the $\frac{f_j}{f_i}$ is a regular function on U_i and hence $f|_{U_i}$ is a morphism. □

Example 2.35. The map

$$f : \mathbb{P}^n - \{(1 : 0 : \cdots : 0)\} \to \mathbb{P}^{n-1}, \quad (x_0 : \cdots : x_n) \mapsto (x_1 : \cdots x_n)$$

is a morphism. ○

Example 2.36. Twisted cubic. ○

quasi-projective alg. sets

2.5. PRODUCTS

We have an obvious set-theoretic identification $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$. However, if we were to take the product topology on the left hand side, then this is not a homeomorphism (on the left hand side all closed sets are finite, while on the right side e.g. $Z(x)$ is not).

Definition 2.37. Let **C** be any category and let X_1 and X_2 be two objects of **C**. An object X is called a product of X_1 and X_2 if it satisfies the following universal property: there exist morphisms $\pi_1 : X \to X_1$, $\pi_2 : X \to X_2$ such that for every object Y and pair of morphisms $f_1 : Y \to X_1, f_2 : Y \to X_2$ there exists a unique morphism $f : Y \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
Y \arrow[dr, dashed, "f"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\
& X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X_2 \\
& X_1 &
\end{tikzcd}$$

As usual, the product is unique up to unique isomorphism (if it exists), and is denoted $X_1 \times X_2$.

Proposition 2.38. *The category of quasi-projective algebraic sets contains all finite products, i.e. if X_1 and X_2 are quasi-projective, then they have a product $X_1 \times X_2$ which is again a quasi-projective algebraic set.*

Sketch of proof. To start, we have $\mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{m+n}$, where the π_1 are just the projections on the corresponding coordinates. If we are given $f_1 : Y \to \mathbb{A}^m$ and $f_2 : Y \to \mathbb{A}^n$ are morphisms, then $f(y) = (f_1(y), f_2(y))$ is again a morphism.

Next, we need to make $\mathbb{P}^m \times \mathbb{P}^n$ into an algebraic set. For this let $U_i \subseteq \mathbb{P}^m$ and $V_j \subseteq \mathbb{P}^n$ be the standard affine open subsets. Then the sets $U_i \times U_j$ cover $\mathbb{P}^m \times \mathbb{P}^n$ and give it the structure of an algebraic set.

To show that $\mathbb{P}^m \times \mathbb{P}^n$ is quasi-projective, we exhibit it as a closed subset of some projective space. For this consider the map (of vector spaces) $k^{m+1} \times k^{n+1} \to k^{m+1} \otimes k^{n+1}$. This is map is compatible with scaling on both sides, so descends to a map $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}(k^{m+1} \otimes k^{n+1}) = \mathbb{P}^{mn+m+n}$, called the *Segre embedding*. Explicitly, the Segre map is given by

$$((x_0 : \cdots : x_m), (y_0 : \cdots : y_m)) \mapsto (\text{point with projective coordinates } x_i y_j).$$

As in the homework one checks that this is a bijection onto a closed subset of $\mathbb{P}^{mn+m+n}$. One also checks (locally) that the inverse is a morphism.

If X_1 and X_2 are quasi-affine, say $X_1 = Z(\mathfrak{a}_1) - Z(\mathfrak{b}_1) \subseteq \mathbb{A}^m$ and $X_2 = Z(\mathfrak{a}_2) - Z(\mathfrak{b}_2) \subseteq \mathbb{A}^m$, then we realize $X_1 \times X_2$ in $\mathbb{A}^m \times \mathbb{A}^n$ as

$$Z(\mathfrak{a}_1^{(x_1,\dots,x_m)}, \mathfrak{a}_2^{(x_{m+1},\dots,x_n)}) - (Z(\mathfrak{b}_1^{(x_1,\dots,x_m)}) \cup Z(\mathfrak{b}_2^{(x_{m+1},\dots,x_n)})),$$

and similarly for quasi-projective algebraic sets. □

Example 2.39. $\mathbb{P}^1 \times \mathbb{P}^1$ is identified with a quadric surface in $\mathbb{P}^3$. [Picture: lines on a hyperbolic parabolid.] ○

By the universal property of the product, we always get a morphism $\Delta : X \to X \times X$, called the *diagonal morphism*:

$$\begin{tikzcd}
X \arrow[drr, bend left, "\mathrm{Id}"] \arrow[dr, dashed, "\Delta"] \arrow[ddr, bend right, "\mathrm{Id}"'] & & \\
& X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \\
& X &
\end{tikzcd}$$

Definition 2.40. A morphism $\varphi : X \to Y$ is called an *immersion* if $\operatorname{im} \varphi \subseteq Y$ is locally closed and the induced morphism $X \to \operatorname{im} \varphi$ is an isomorphism. If $\operatorname{im} \varphi$ is open (resp. closed) then we call φ an *open* (resp. *closed*) *immersion*.

Remark 2.41. A locally closed subset of a quasi-projective algebraic set is a quasi-projective algebraic set.

Proposition 2.42. *Let X be a quasi-projective algebraic set. Then $\Delta : X \to X \times X$ is a closed immersion. We say that X is* separated.

Remark 2.43. Recall that that a topological space Y is Hausdorff if and only if the diagonal is a closed subspace of $Y \times Y$ with the product topology (!). Thus Proposition 2.42 says that at least in some ways quasi-projective algebraic sets behave like Hausdorff spaces.

Remark 2.44. A general algebraic space need not be separated (e.g. affine line with doubled origin).

Proof. Let $X \subseteq \mathbb{P}^n$ be a quasi-projective algebraic set. Then $\Delta(X) = (X \times X) \cap \Delta_{\mathbb{P}^n} \subseteq \mathbb{P}^n \times \mathbb{P}^n$. So it is enough to check the statement for $X = \mathbb{P}^n$.

The diagonal $\Delta_{\mathbb{P}^n}$ consists of pairs of points $((x_0 : \cdots : x_n), (y_0 : \cdots : y_n))$ such that $(x_0 : \cdots : x_n) = (y_0 : \cdots : y_n)$, i.e. such that the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ y_0 & y_1 & \cdots & y_n \end{pmatrix}$$

has rank 1. Thus

$$\Delta_{\mathbb{P}^n} = \{((x_0 : \cdots : x_n), (y_0 : \cdots : y_n)) : x_i y_j - x_j y_i = 0 \text{ for all } i, j\}$$

is closed in $\mathbb{P}^n \times \mathbb{P}^n$ (in Segre coordinates $z_{ij} = x_i y_j$ it is given by $z_{ij} - z_{ji} = 0$). $\square$

Corollary 2.45. *Let $f, g\colon X \to Y$ be two morphisms of quasi-projective algebraic sets. Suppose that there exists a dense open subset U such that $f|_U = g|_U$. Show that $f = g$ on X.*

Proof. Exercise. $\square$

Remark 2.46. Let X be the affine line with doubled origin and let $f, g\colon \mathbb{A}^1 \to X$ be the two copies of $\mathbb{A}^1$. Then $f = g$ on $\mathbb{A}^1 - \{0\}$, but $f \neq g$ at 0.

Definition 2.47. Let **C** be any category and let $f_1 : X_1 \to Z$ and $f_2 : X_2 \to Z$ be two morphisms of **C**. An object X is called a *fiber product product of X_1 and X_2 over Z*if it satisfies the following universal property: there exist morphisms $\pi_1 : X \to X_1$, $\pi_2 : X \to X_2$ such that for every object Y and pair of morphisms $g_1 : Y \to X_1$, $g_2 : Y \to X_2$ such that $f_1 \circ g_1 = f_2 \circ g_2$ there exists a unique morphism $h\colon Y \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
Y \arrow[dr, dashed, "h"] \arrow[drr, bend left, "g_2"] \arrow[ddr, bend right, "g_1"'] & & \\
& X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X_2 \arrow[d, "f_2"] \\
& X_1 \arrow[r, "f_1"] & Z
\end{tikzcd}$$

The fiber product is unique up to unique isomorphism (if it exists), and is denoted $X_1 \times_Z X_2 = X_1 \times_{f_1,Z,f_2} X_2$. A commutative diagram of the form

$$\begin{tikzcd} X_1 \times_Z X_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"] & Z \end{tikzcd}$$

is called a *Cartesian square*.

Example: fiber product in **Set**.

Proposition 2.48. *The category of quasi-projective algebraic sets admits fiber products.*

Proof. Let $f_i : X_i \to Z$. Set $X_1 \times_Z X_2 = (f_1 \times f_2)^{-1}(\Delta_Z) \subseteq X_1 \times X_2$.

$$\begin{tikzcd} (f_1 \times f_2)^{-1}(\Delta_Z) \arrow[r, hook, "\text{closed}"] \arrow[d] & X_1 \times X_2 \arrow[d, "f_1 \times f_2"] \\ Z \arrow[r, hook, "\text{closed}", "\Delta_Z"'] & Z \times Z. \end{tikzcd}$$

This is a closed subset of $X_1 \times X_2$ and hence a quasi-projective algebraic set. One quickly checks the universal property. □

Exercise 2.49. Let $f : X \to Y$ and let $Z \subseteq X$ be a locally closed subset. Then $f^{-1}(Z) = Z \times_Y X$.

Definition 2.50. Let **C** be any category and let X_1 and X_2 be two objects of **C**. An object X is called a *coproduct* of X_1 and X_2 if it satisfies the following universal property: there exist morphisms $i_1 : X_1 \to X_2$, $i_2 : X_2 \to X$ such that for every object Y and pair of morphisms $f_1 : X_1 \to Y$, $f_2 : X_2 \to Y$ there exists a unique morphism $f : X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} & X_1 \arrow[d, "i_1"] \arrow[ddr, bend left, "f_1"] & \\ X_2 \arrow[r, "i_2"] \arrow[drr, bend right, "f_2"'] & X \arrow[dr, dashed, "f"] & \\ & & Y \end{tikzcd}$$

The coproduct is unique up to unique isomorphism (if it exists).

Example: **Set**, **Mod**(R).

Proposition 2.51. *The category of quasi-projective algebraic sets admits coproducts.*

Proof. Let $X \subseteq \mathbb{P}^m$ and $Y \subseteq \mathbb{P}^n$. We need to identify the disjoint union of X and Y. Choose $P \in \mathbb{P}^m - X$ and $Q \in \mathbb{P}^n - Y$ (if necessary embed into larger $\mathbb{P}^k$ first). Then $X \sqcup Y = X \times \{Q\} \cup \{P\} \times Y \subseteq \mathbb{P}^m \times \mathbb{P}^n$. ☐

2.6. Example: the Grassmannian

The Grassmannian of r-planes in n-spaces is, as a set,

$$\mathrm{Grass}(r,n) = \{V \subseteq k^n : V \text{ is an } r\text{-dimensional subspace}\}.$$

For example, $\mathrm{Grass}(1,n) = \mathbb{P}^n$. We want to give the Grassmannian the structure of an algebraic set. Given $V \subseteq k^n$, we could try to choose a basis $f_1, \dots, f_r$ of V and use that to assign coordinates of V. However, we then have to deal with that fact, that we cannot canonically choose a basis of V.

To solve this problem, we need to look at the exterior product $\bigwedge^r k^n$. Recall that $\bigwedge^r k^n$ contains sums of elements of the form $v_1 \wedge \cdots \wedge v_r$ with $v_i \in k^n$ such that $v_1 \wedge \cdots \wedge v_i \wedge v_{i+1} \wedge \cdots \wedge v_r = -v_1 \wedge \cdots \wedge v_{i+1} \wedge v_i \wedge \cdots \wedge v_r$.

Now take a basis $f_1, \dots, f_r$ of $V \subseteq k^n$ and send it to $f_1 \wedge \cdots \wedge f_r \in \bigwedge^r k^n$. Then if we take a different basis $f_1', \dots, f_r'$ of V, we have $f_1 \wedge \cdots \wedge f_r = \alpha f_1' \wedge \cdots \wedge f_r'$, where $\alpha \neq 0$ is the determinant of the change-of-basis transformation. Thus V determines a line in $\bigwedge^r k^n$. So we get an injective map

$$P\colon \mathrm{Grass}(r,n) \to \mathbb{P}(\textstyle\bigwedge^r k^n) = \mathbb{P}^{\binom{n}{r}-1}.$$

This map is called the *Plücker embedding*.

We want to show that the image of P is a closed subset. For this note, that $\operatorname{im} P$ consists exactly of the pure tensors, i.e. elements that can be written as $v_1 \wedge \cdots \wedge v_r$ (as opposed to sums of these elements).

Lemma 2.52. *Let $r < n$ and fix a non-zero $\omega \in \bigwedge^r k^n$. Then the linear map*

$$f : k^n \to \textstyle\bigwedge^{r+1} k^n, \qquad v \mapsto v \wedge \omega$$

has $\operatorname{rank} f \geq n - r$ *with equality holding if and only if* $\omega = v_1 \wedge \cdots \wedge v_r$ *for some* $v_1, \dots, v_r \in k^n$.

Proof. Let $K = \dim \ker f = n - \operatorname{rank} f$. Choose a basis $v_1, \dots, v_K$ of $\ker f$ and extend it to a basis $v_1, \dots, v_n$ of k^n. Express ω in the corresponding basis of $\bigwedge^r k^n$:

$$\omega = \sum_{i_1 < \cdots < i_r} a_{i_1 \dots i_r} v_{i_1} \wedge \cdots \wedge v_{i_r}.$$

Then for $i \in \{1, \dots, K\}$ we have $v_i \in \ker f$ and hence

$$0 = v_i \wedge \omega = \sum_{i_1 < \cdots < i_r} a_{i_1 \dots i_r} v_i \wedge v_{i_1} \wedge \cdots \wedge v_{i_k} = \sum_{\substack{i_1 < \cdots < i_r \\ i \notin \{i_1, \dots, i_r\}}} a_{i_1 \dots i_r} v_i \wedge v_{i_1} \wedge \cdots \wedge v_{i_r}.$$

Since the vectors $v_i \wedge v_{i_1} \wedge \cdots \wedge v_{i_k}$ for $i \notin \{i_1, \dots, i_k\}$ are part of a basis of $\bigwedge^{r+1} k^n$ we must have $a_{i_1 \dots i_r} = 0$ in these cases. This works for all $i = 1, \dots, K$, so that $a_{i_1 \dots i_r}$ can only be non-zero if $\{1, \dots, K\} \subseteq \{i_1, \dots, i_r\}$.

On the other hand, since $\omega \neq 0$, at least one of the coefficients has to be non-zero. In particular, this requires that $K \leq r$ and hence $\operatorname{rank} f = n - K \geq n - r$. Moreover, if we have equality, then only the coefficient $a_{1 \dots K}$ can be non-zero, which means that ω is a scalar multiple of $v_1 \wedge \cdots \wedge v_r$.

Conversely, if $\omega = w_1 \wedge \cdots \wedge w_r$ for some (necessarily linearly independent) $w_i \in k^n$, then $w_1, \dots, w_r \in \ker f$ and hence $\dim \ker f \geq r$, i.e. $\operatorname{rank} f \leq n - r$. □

Theorem 2.53. $\operatorname{Grass}(r, n)$ *is a projective algebraic set.*

Proof. This is clearly true for $\operatorname{Grass}(n, n)$, which is just a point. So we can assume that $r < n$. We have to show that the image of the Plücker map is a closed subset of $\mathbb{P}^{\binom{n}{r}-1}$. By construction, a point of $\omega \in \mathbb{P}^{\binom{n}{r}-1}$ lies in (the image of) $\operatorname{Grass}(r, n)$ if and only if it is the class of a pure tensor $v_1 \wedge \cdots \wedge v_r$. By the lemma, this is the case if and only if the rank of $f : k^n \to \bigwedge^{r+1} k^n$, $v \mapsto v \wedge \omega$ is $n - r$. Since we also know that the rank is at least $n - r$, this can be checked by the vanishing of all $(n - r + 1) \times (n - r + 1)$ minors of the matrix corresponding to f. But the entries of that matrix are polynomials in the coordinates of ω and the minors are polynomials in the entries of the matrix. Thus all such ω form a closed subset of $\mathbb{P}^{\binom{n}{r}-1}$. □

Remark: The Grassmannian as homogeneous space.

2.7. "COMPACTNESS"

The most important property of real or complex projective space in the analytic (i.e. "usual") topology is that they are compact. We already noted that all quasi-projective algebraic sets are (quasi-)compact. But since they are not Hausdorff, in many ways they don't behave the way we expect them to. For example, one of the main properties of compact Hausdorff spaces is that closed subsets are again compact and compact subsets are mapped to compact subsets by continuous maps. But as we have seen in the example of the projection map $\{xy = 1\} \to \mathbb{A}^1$ this is not true for a general algebraic set.

Definition 2.54. A map $f : X \to Y$ between topological spaces is called *closed* if for every closed subset $Z \subseteq X$, the image $f(Z) \subseteq Y$ is closed.

Definition 2.55. Let X be a quasi-projective algebraic set. Then X is called *complete* if for every quasi-projective algebraic set Y the projection map $\pi : X \times Y \to Y$ is closed.

Proposition 2.56. *Let X be a complete quasi-projective algebraic set and let $f : X \to Y$ be any morphism. Then f is closed.*

Proof. Consider the graph

$$\Gamma_f = \{(x, f(x)) \in X \times Y\}.$$

Then $\Gamma_f = (f \times \mathrm{Id})^{-1}(\Delta_Y)$ is closed and we can factor f as $\pi \circ i$, where $i\colon X \to X \times Y, x \mapsto (x, f(x))$ is a closed immersion onto Γ_f and $\pi\colon X \times Y \to Y$ is the projection map. If $Z \subseteq X$ is closed, then its image $i(Z)$ is closed in Γ_f and hence in $X \times Y$. Thus $f(Z) = \pi(i(Z))$ is closed. □

Remark 2.57. The technique of factoring a map in this way as a closed immersion (which is proper) followed by a projection (which is smooth if X is) is useful in many situations.

Theorem 2.58. *Every projective algebraic set is complete.*

Example: twisted cubic

The hard work for this theorem is contained in the following special case.

Lemma 2.59 (Main theorem of elimination theory)**.** *The projection map $\pi\colon \mathbb{P}^m \times \mathbb{A}^n \to \mathbb{A}^n$ is closed.*

Proof. Let $Z \subseteq \mathbb{P}^m \times \mathbb{A}^n$ be closed, defined by an ideal $\mathfrak{a} \subseteq k[x_0, \dots, x_m; y_1, \dots, y_n]$, homogeneous in the variables $x_0, \dots, x_m$.

We will show that if $(a_1, \dots, a_n) \notin \pi(Z)$, then there exists some $f \in k[y_1, \dots, y_n]$ such that $D(f)$ is a open neighborhood of $(a_1, \dots, a_n)$ in $\mathbb{A}^n - \pi(Z)$, i.e. such that

- $f(a_1, \dots, a_n) \neq 0$, and
- whenever $f(b_1, \dots, b_n) \neq 0$, then $(b_1, \dots, b_n) \notin \pi(Z)$.

Let

$$\mathrm{ev}_{a_1,\dots,a_n}\colon k[x_0, \dots, x_m; y_1, \dots, y_n] \to k[x_0, \dots, x_m], \quad f \mapsto f(x_0, \dots, x_m, a_1, \dots, a_n)$$

be the evaluation map. Then

$$\begin{aligned}
(a_1, \dots, a_n) \notin \pi(Z) &\Leftrightarrow \widetilde{Z}(\mathrm{ev}_{a_1,\dots,a_n}(\mathfrak{a})) = \emptyset \text{ in } \mathbb{P}^n \\
&\Leftrightarrow Z(\mathrm{ev}_{a_1,\dots,a_n}(\mathfrak{a})) \subseteq \{(0, \dots, 0)\} \text{ in } \mathbb{A}^{n+1} \\
&\Leftrightarrow \sqrt{\mathrm{ev}_{a_1,\dots,a_n}(\mathfrak{a})} \supseteq (x_0, \dots, x_m) \\
&\Leftrightarrow \mathrm{ev}_{a_1,\dots,a_n}(\mathfrak{a}) \supseteq (x_0, \dots, x_m)^N \text{ for } N \gg 0 \\
&\Leftrightarrow \mathrm{ev}_{a_1,\dots,a_n}(\mathfrak{a})_N = k[x_0, \dots, x_m]_N \text{ for some } N \gg 0.
\end{aligned}$$

(Here we use the notation $S_N = S \cap k[x_0, \dots, x_m]_N$ for $S \subseteq k[x_0, \dots, x_m]$.) Letting $\{f_i\}$ be a k-basis for $k[x_0, \dots, x_m]_N$ we can find $g_i \in \mathfrak{a}_N$ such that $\mathrm{ev}_{a_1,\dots,a_n}(g_i) = f_i$. Since $\{f_i\}$, form a basis of $k[x_0, \dots, x_m]_N$, we can write each g_j as a linear combination of the f_i over $k[y_1, \dots, y_n]$. In other words, there exists a matrix M with entries in $k[y_1, \dots, y_n]$ such that

$$\begin{pmatrix} g_1 \\ \vdots \\ g_{\binom{m+N}{N}} \end{pmatrix} = M \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_{\binom{m+N}{N}} \end{pmatrix}.$$

Clearly $\mathrm{ev}_{a_1,\dots,a_n}(M) = \mathrm{Id}$. Let $f = \det M$. Then $f(a_1,\dots,a_n) = 1 \neq 0$ as required. Further, if $f(b_1,\dots,b_n) \neq 0$, then $\mathrm{ev}_{b_1,\dots,b_n}(M)$ is invertible. Therefore the $\mathrm{ev}_{b_1,\dots,b_n}(g_i)$ form a basis of $k[x_0,\dots,x_m]_N$ and hence $\mathrm{ev}_{b_1,\dots,b_n}(\mathfrak{a})_N = k[x_0,\dots,x_m]_N$. Reading the equivalences backwards we see that then $(b_1,\dots,b_n) \notin \pi(Z)$. □

Note on elimination theory.

Proof of Theorem 2.58. Clearly every closed subset of a complete algebraic set is complete. Thus it suffices to check that $\mathbb{P}^n$ is complete. Let Y be any quasi-projective algebraic set and let $\pi\colon \mathbb{P}^n \times Y \to Y$ be the projection map. Then Y can be covered by finitely many affine algebraic subsets U_i. If $\mathbb{P}^n \times U_i \to U_i$ is closed for all i, the π is closed. So we can assume that Y is affine, i.e. it is a closed subspace of some $\mathbb{A}^m$. Thus the statement follows from Lemma 2.59. □

In algebraic geometry it is often important to have relative versions of properties. The relative version of "complete" is the following definition.

Definition 2.60. A morphism $f\colon X \to Y$ of quasi-projective algebraic sets is called *proper* if it is universally closed[2], i.e. if for every morphism $g\colon Z \to Y$ (with Z quasi-projective) the pullback $Z \times_Y X \to Z$ is closed.

Example 2.61. A quasi-projective algebraic set X is complete if and only if the map $X \to \mathrm{pt}$ is proper (note that $Z \times_{\mathrm{pt}} X = Z \times X$). ○

Theorem 2.62. *Let $f\colon X \to Y$ be a morphism of quasi-projective algebraic sets. Then the following are equivalent:*

(i) f is proper.

(ii) For every quasi-projective algebraic set Z the map $Z \times X \to Z \times Y$ is closed.

(iii) f is projective*, i.e. f factors as*

$$\begin{tikzcd} X \arrow[r, hook, "\text{closed}"] \arrow[dr, "f"'] & Y \times \mathbb{P}^n \arrow[d, "\mathrm{pr}_1"] \\ & Y \end{tikzcd}$$

(iv) Each point $y \in Y$ has an open neighborhood V such that $f|_{f^{-1}(y)}\colon f^{-1}(V) \to V$ factors as

$$\begin{tikzcd} f^{-1}(V) \arrow[r, hook, "\text{closed}"] \arrow[dr, "f"'] & V \times \mathbb{P}^n \arrow[d, "\mathrm{pr}_1"] \\ & V \end{tikzcd}$$

[2]The reason for introducing the extra word "proper" is that in the non-quasi-projective case we need to also require that f is separated and of finite type.

Corollary 2.63. *Every complete quasi-projective algebraic set is projective.*

Remark 2.64. Theorem 2.62 and Corollary 2.63 do not in general hold in the non-quasi-projective case (though counterexamples are extremely hard to find).

Proof of Theorem 2.62. Let f be proper and consider the map $Z \times Y \to Y$. Then $(Z \times Y) \times_Y X = Y \times X$, so that (i) implies (ii).

Conversely, assuming (ii), let $g\colon Z \to Y$ be a morphism. Then, as we have seen before, $\Gamma_g\colon Z \hookrightarrow Z \times Y$ is closed. Thus the commutative diagram

$$\begin{tikzcd} Z \times_Y X \arrow[r, hook, "\text{closed}"] \arrow[d] & Z \times X \arrow[d, "\text{closed}"] \\ Z \arrow[r, hook, "\Gamma_g"] & Z \times Y \end{tikzcd}$$

shows (i).

Clearly (iii) implies (iv).

Next we will show that (iv) implies (ii). A map being closed is local on the base, so for checking (ii) we may replace $X \to Y$ by $f^{-1}(V_i) \to V_i$ for an open cover $\{V_i\}$ of Y realizing (iv). Thus we may assume that we have a factorization as in (iii) (note that we are not saying here that (iv) implies (iii)). But then the map $Z \times X \to Z \times Y$ factors as

$$\begin{tikzcd} Z \times X \arrow[r, hook, "\text{closed}"] \arrow[dr] & Z \times Y \times \mathbb{P}^n \arrow[d, "\mathrm{pr}_{Z\times X}"] \\ & Z \times Y \end{tikzcd}$$

and hence is closed because $\mathbb{P}^n$ is complete.

Finally, we show that (ii) implies (iii). As X is quasi-projective, there exists an immersion $i\colon X \hookrightarrow \mathbb{P}^n$. By assumption, the map $f \times \mathrm{Id}\colon X \times \mathbb{P}^n \to Y \times \mathbb{P}^n$ is closed. So

$$\begin{tikzcd} X \arrow[r, hook, "\Gamma_i"] \arrow[drr, "f"'] & X \times \mathbb{P}^n \arrow[r, "f\times\mathrm{Id}"] & Y \times \mathbb{P}^n \arrow[d, "\mathrm{pr}_1"] \\ & & Y \end{tikzcd}$$

gives the desired factorization, where we note that $f \times \mathrm{Id}$ is injective on the image of Γ_i. $\square$

Proposition 2.65. *The following holds (even without assuming quasi-projectiveness):*

(i) Closed immersions are proper.

(ii) Projective morphisms are proper.

(iii) *A composition of proper morphisms in proper.*

(iv) *Proper morphisms are stable under base change, i.e. if $f: X \to Y$ is proper and $g: Z \to Y$ is any morphism, then $Z \times_Y X \to Z$ is proper.*

(v) *Properness is local on the base.*

Proof. In the case of quasi-projective algebraic sets this is an exercise. For the general case, see [H, Section II.4]. □

3. SOME CONSTRUCTIONS, DEFINITIONS AND EXAMPLES

3.1. SINGULARITIES

Definition 3.1. Let $X = Z(f_1, \dots, f_r) \subseteq \mathbb{A}^n$ be an irreducible affine algebraic set. Then X is *nonsingular at a point $a \in X$* if the rank of the *Jacobian matrix*

$$\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{\substack{1\le i\le r\\ 1\le j\le n}}$$

is $n - \dim X$. Further, X is *nonsingular*, if it is nonsingular at every point.

Synonyms for “nonsingular” are “smooth” and “regular”.

Example 3.2. (in characteristic 0) $y^2 = x^3$ (cusp), $y^2 = x^3 + x^2$ (node), $y^2 = x^3 + x^2 + 1$ (smooth) ○

Note on differentiation in positive characteristic.

There are two problems with this definition: It is not immediately evident that it is independent of the embedding of X into affine space; and it does not directly generalize to arbitrary quasi-projective algebraic sets.

Theorem 3.3. *Let $X \subseteq \mathbb{A}^n$ be an irreducible affine algebraic set and $a \in X$ a point. Then Y is nonsingular at a if and only if $(\mathcal{O}_X)_a$ is a regular local field.*

Recall from commutative algebra, that a noetherian local ring A with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$ is regular if $\dim_k \mathfrak{m}^2/\mathfrak{m} = \dim A$. Also recall that always $\dim_k \mathfrak{m}^2/\mathfrak{m} \geq \dim A$.

In our case $A = \mathcal{O}_{X,a}$ is the stalk of the structure sheaf at a, which is the localization of $\mathcal{O}_X(X)$ at the maximal ideal $\mathfrak{m}_a = \{f \in I(X) : f(a) = 0\}$. We note that the residue field of $\mathcal{O}_{X,a}$ is the ground field k.

Lemma 3.4. *Let X be an irreducible algebraic set and $a \in X$ a point. Then* $\dim O_{X,a} = \dim X$.

Proof. It suffices to prove the statement for affine algebraic sets. Then by the Nullstellensatz $\dim \mathcal{O}_X(X) = \dim X$ and $\mathcal{O}_X(X)$ is an integral domain. We also know that $\mathcal{O}_{X,a}$ is the localization of $\mathcal{O}_X(X)$ at the maximal ideal $\mathfrak{m}_a$. The statement now follows from the fact that each maximal ideal in an integral finite type k-algebra has the same height [E, Theorem A on p. 286]. □

Proof of Theorem 3.3. Let $a = (a_1, \dots, a_n) \in \mathbb{A}^n$ and let $\mathfrak{q}_a = (x_1 - a_1, \dots, x_n - a_n) \lhd k[x_1, \dots, x_n]$ be the corresponding maximal ideal. We consider the linear map

$$\theta\colon k[x_1, \dots, x_n] \to k^n, \quad f \mapsto \left(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a)\right).$$

Then $\theta(x_i - a_i) = e_i$ form a basis of k^n and $\theta(\mathfrak{q}_a^2) = 0$. Thus θ induces an isomorphism of k-vector spaces $\theta'\colon \mathfrak{q}_a/\mathfrak{q}_a^2 \xrightarrow{\sim} k^n$.

Now let $I(X) = (f_1, \dots, f_r)$ and let $J = \left(\frac{\partial f_i}{\partial x_j}(a)\right)$ be the Jacobian matrix. Then the rank of J is just the dimension of $\theta(I(X))$ as a subspace of k^n. Under θ' this is also the same as the subspace $(I(X) + \mathfrak{q}_a^2)/\mathfrak{q}_a^2$ of $\mathfrak{q}_a/\mathfrak{q}_a^2$ (note that $I(X) \subseteq \mathfrak{q}_a$).

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,a}$. The ring $\mathcal{O}_{X,a}$ is obtained from $k[x_1, \dots, x_n]$ by dividing by $I(X)$ and localizing at $\mathfrak{q}_a$. Thus $\mathfrak{m}/\mathfrak{m}^2 \cong \mathfrak{q}_a/(I(X) + \mathfrak{q}_a^2)$.

To summarize:

$$\begin{aligned}\dim_k \mathfrak{q}_a/(I(X) + \mathfrak{q}_a^2) &= \dim_k \mathfrak{m}/\mathfrak{m}^2,\\ \dim_k (I(X) + \mathfrak{q}_a^2)/\mathfrak{q}_a^2 &= \operatorname{rank} J,\\ \dim_k \mathfrak{q}_a/\mathfrak{q}_a^2 &= n.\end{aligned}$$

Thus $\dim_k \mathfrak{m}/\mathfrak{m}^2 + \operatorname{rank} J = n$.

Now, $\mathcal{O}_{X,a}$ is regular if and only if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim \mathcal{O}_{X,a}$ and the latter is equal to $\dim X$ by Lemma 3.4. But by what we showed above, this is equivalent to to $\operatorname{rank} J = n - \dim X$. □

Definition 3.5. Let X be any algebraic set. Then X is *nonsingular* (or *regular* or *smooth*) *at the point* $a \in X$ if the local ring $\mathcal{O}_{X,a}$ is regular. Further, X is *nonsingular* (*regular, smooth*) if it is nonsingular at every point and it it *singular* if it is not non-singular.

Remarks 3.6.

- Thus non-singular is a local property, i.e. we can check it by checking on each open subset of a cover.
- Every regular local ring is an integral domain (Auslander-Buchsbaum theorem). Thus an algebraic set is locally irreducible around each smooth point, i.e. there is only one irreducible component passing through the point. Conversely, any point where two or more irreducible components meet is necessarily singular [picture]. As a consequence, in order to study smoothness phenomena, we can restrict our attention to irreducible algebraic sets.
- For a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k we always have $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$. Thus by the proof of the theorem, the rank of the Jacobian is always at most $n - \dim X$.

Definition 3.7. For an algebraic set X, let $\operatorname{Sing}(X)$ be the subset of singular points of X.

Lemma 3.8. *Let X be an irreducible algebraic set. Then* $\mathrm{Sing}(X)$ *is a closed subset of X*

Proof. Covering X by affine subsets, it suffices to show the statement for affine X. By the remark above it suffices to show that the locus of points where $\operatorname{rank} J < n - \dim X$ is closed. But that locus is given by the vanishing of all the $(n - \dim X) \times (n - \dim X)$ minors of J and hence is a closed subset. □

Proposition 3.9. *Let X be an irreducible quasi-projective algebraic set. Then the smooth points of X form a dense open subset.*

Proof. We will only show this for $X = Z(f)$ is a hypersurface in $\mathbb{A}^n$ given by a single irreducible polynomial. The general case can be reduced to this, but we do not yet have the technology to do so available.

We already know that $\mathrm{Sing}(X)$ is closed, so we only have to show that $\mathrm{Sing}(X) \neq X$. Now, the set $\mathrm{Sing}(X)$ consists exactly of the points $a \in X$ such that $\frac{\partial f}{\partial x_i}(a) = 0$ for $i = 1, \ldots, n$. But then $\frac{\partial f}{\partial x_i}$ vanish on X and hence are contained in $I(X) = (f)$, i.e. they are divisible by f But $\deg \frac{\partial f}{\partial x_i} \leq \deg f - 1$, so we must have $\frac{\partial f}{\partial x_i} = 0$ for all i.

If $\operatorname{char} k = 0$, this already implies that f is a constant, so either $X = \mathbb{A}^n$ or $X = \emptyset$ and we are done. In characteristic p, $\frac{\partial f}{\partial x_i} = 0$ implies that f is actually a polynomial in x_i^p. But then we can get a polynomial g such that $f = g^p$ (using that k is algebraically closed and we can take pth roots of the coefficients of f). This is a contradiction to f being irreducible. □

We will see some more examples of singularities in the homework. For now we will finish this section with the main theorem on the theory of singularities.

Theorem 3.10 (Hironaka, 1964)**.** *Let* $\operatorname{char} k = 0$ *and let X be an irreducible quasi-projective algebraic set. Then there exists a smooth quasi-projective algebraic set $\tilde{X}$ and a proper morphism $\pi : \tilde{X} \to X$ such that π induces an isomorphism of open subsets* $\pi^{-1}(X - \mathrm{Sing}(X)) \to X - \mathrm{Sing}(X)$.

A morphism π as in the theorem in called a *resolution of singularities of X*. It is still an open problem whether resolutions of singularities always exist in positive characteristic.

3.2. RATIONAL MAPS

[Intro: resolution of singularities; classification problem; restrict to irreducibles; recall: morphisms that agree on an open subset agree everywhere]

Definition 3.11. Let X and Y be irreducible quasi-projective algebraic sets.

- A *rational map* $f : X \dashrightarrow Y$ is an equivalence class of pairs (U, f_U), where $U \subseteq X$ is a non-empty open subset and $f_U : U \to Y$ is a morphism, where $(U, f_U) \sim (V, f_V)$ if $f_U|_{U \cap V} = f_V|_{U \cap V}$. (Recall that in an irreducible quasi-projective algebraic set every non-empty open subset is dense.)

- A rational map $f: X \dashrightarrow Y$ is *dominant* if the image of some (and hence every) representative contains a non-empty open subset.
- Dominant rational maps can be composed by composing representatives. Thus we can form a category of irreducible quasi-projective varieties and dominant rational maps between them.
- A rational map $f: X \dashrightarrow Y$ is called *birational* if it is an isomorphism in this category, i.e. if it is dominant and there exists a rational map $g: Y \dashrightarrow X$ such that $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$.
- X and Y are *birational* if there is a birational map $f: X \dashrightarrow Y$ between them.
- A *birational morphism* is a dominant morphism $f: X \to Y$, which has an inverse as rational map, i.e. it is a birational map that is defined everywhere.

Example 3.12. A resolution of singularities is a birational morphism. ○

The subject of *birational geometry* is to study properties of algebraic sets that are invariant under birational maps. In particular, one wants to classify all birational equivalence classes of algebraic sets (keyword: minimal model program).

Definition 3.13. Let X be an irreducible quasi-projective algebraic set. A rational map $\varphi: X \dashrightarrow \mathbb{A}^1$ is called a *rational function*. In other words a rational function is given by a regular function $\varphi \in \mathcal{O}_X(U)$ for some non-empty open subset U.

The set of all rational functions on X is denoted by $K(X)$ and called the *function field* of X. It is indeed a field: addition and multiplication can be defined in the obvious way (on the intersection of the sets of definition). The inverse of $0 \neq \varphi \in \mathcal{O}_X(U)$ is well defined on $U - Z(\varphi)$.

Exercise 3.14. Let X be an irreducible affine algebraic set. Show that the function field $K(X)$ is isomorphic to the quotient field of $A(X)$. Also show that every local ring $\mathcal{O}_{X,a}$ is naturally a subring of $K(X)$.

Lemma 3.15. *The dimension of an irreducible quasi-projective algebraic set X is equal to the transcendence degree of $K(X)$ over k.*

Proof. Let $U \subseteq X$ be an open affine subset. Then $\dim X = \dim U = \dim A(U)$. By a standard result of commutative algebra, this is in turn the same as the transcendence degree of $K(U) = K(X)$ over k. □

If $f: X \dashrightarrow Y$ is a dominant rational map, we get a k-algebra homomorphism $f^*: k(Y) \to k(X)$, by $f^*\varphi = \varphi \circ f$

Proposition 3.16. *This gives a contravariant equivalence of categories*

$$\left\{\begin{matrix} \textit{irreducible quasi-projective algebraic sets} \\ + \\ \textit{dominant rational maps} \end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix} \textit{finitely generated field extensions of } k \\ + \\ k\textit{-algebra homomorphisms} \end{matrix}\right\}.$$

Proof. Since $k(X) = k(U)$ for any open subset U of X, we really only need to consider irreducible affine algebraic sets. There the construction is very similar to the equivalence between affine algebraic sets and finite type reduced k-algebras. We refer to [H, Theorem I.4.4] for details. □

Corollary 3.17. *For any two irreducible quasi-projective algebraic sets X and Y the following are equivalent:*

(i) X and Y are birationally equivalent;

(ii) there are open subsets $U \subseteq X$ any $V \subseteq Y$ with $U \cong V$;

(iii) $K(X) \cong K(Y)$ as k-algebras.

We are often interested in *generic properties* of algebraic sets, i.e. properties that are true on a dense open subset. For example, we (partially) showed earlier that every irreducible quasi-projective algebraic set is generically smooth: the smooth points always form a dense open subset. The following proposition can be quite useful for proving generic properties.

Proposition 3.18. *Any irreducible quasi-projective algebraic set X of dimension d is birational to a hypersurface in $\mathbb{P}^{d+1}$.*

Proof. The function field $K(X)$ is a finitely generated field extension of the algebraically closed (and hence perfect) field k. It has transcendence degree d. Thus we can find a transcendence base $x_1, \dots, x_d \in K(X)$ such that $K(X)$ is a finite separable extension of $k(x_1, \dots, x_d)$. By the theorem of the primitive element we can thus find a further element $y \in K(X)$ such that $K(X) = k(x_1, \dots, x_d, y)$.

Now y is algebraic over $k(x_1, \dots, x_d)$, so it satisfies a polynomial equation with coefficients which are rational functions in $x_1, \dots, x_d$. Clearing denominators, we get an irreducible polynomial over k with $f(x_1, \dots, x_d, y) = 0$. This defines a hypersurface in $\mathbb{A}^{d+1}$ with function field $K(X)$. By the preceding corollary this hypersurface is birational to X and hence so is its closure Y in $\mathbb{P}^{d+1}$. □

We can now finish the proof of Proposition 3.9: To show that $\mathrm{Sing}(X) \neq X$ it suffices to show that $\mathrm{Sing}(U) \neq U$ for some open subset U of X. Thus we can replace X by a birational quasi-projective algebraic set. In particular we can replace it by the hypersurface obtained in the proposition.

3.3. BLOWING UP

Next, we will discuss the most important example of a birational map.

Definition 3.19. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set and let Z be a closed subset with $I(Z) = (f_1, \dots, f_r)$. We obtain a morphism

$$f : X - Z \to \mathbb{P}^{r-1}, \quad x \mapsto (f_1(x) : \cdots : f_r(x)).$$

The graph $\Gamma_f = \{(x, f(x)) : x \in X - Z\}$ is isomorphic to $X - Z$ and closed in $(X - Z) \times \mathbb{P}^{r-1}$. The closure of Γ_f in $X \times \mathbb{P}^{r-1}$ is called the *blow-up of X at Z* and denoted $\tilde{X}$. It contains a dense open subset isomorphic to $X - Z$ and a natural projection morphism $\pi : \tilde{X} \to X$.

Remark 3.20. One can check, that $\tilde{X}$ is independent of the chosen generators of $I(Z)$, c.f. [G2, Lemma 9.16]. It is also possible to define the blowup for arbitrary quasi-projective algebraic sets by gluing (though it does get complicated if Z is not irreducible). We will not prove this here are we are mainly interested in computing some examples of blowups in order to get some intuition for what this construction does. Our examples will usually consist of blowing up at just one point, so that the gluing is trivial (as nothing happens outside of a neighborhood of the point).

Before we consider some examples let us look at some general properties and introduce some names for special parts of the blow-up.

The morphism π gives an isomorphism from $\Gamma_f \subseteq \tilde{X} \subseteq X \times \mathbb{P}^{r-1}$ to $X - Z$. By abuse of notation we will usually call both of these open sets U. On the complement of U the morphism π is usually not an isomorphism.

If X is irreducible and $Z \neq X$ then U is a dense open subset of both X and $\tilde{X}$. Thus $\tilde{X}$ (which is the closure of U) is also irreducible and π is a birational morphism.

Definition 3.21. The closed subset $\tilde{X} - U = \pi^{-1}(Z)$ is called the *exceptional set* of the blow-up.

If Y is a closed subset of X not contained in Z, we can also blow up Y at $Z(f_1, \dots, f_r)$. The resulting space $\tilde{Y} \subseteq Y \times \mathbb{P}^{r-1} \subseteq X \times \mathbb{P}^{r-1}$ is then also a closed subset of $\tilde{X}$. It is actually the closure of $Y \cap U$ in $\tilde{X}$.

Definition 3.22. The subset $\tilde{Y}$ of $\tilde{X}$ is called the *strict transform* of Y.

In order to compute examples, the following lemma is very helpful:

Lemma 3.23. *With the notation from above we have*

$$\tilde{X} \subseteq \{(x, y) \in X \times \mathbb{P}^{r-1} : y_i f_j(x) = y_j f_i(x) \text{ for all } 1 \leq i, j \leq r\}.$$

Proof. Any point (x, y) on the graph Γ_f of $f : U \to \mathbb{P}^{r-1}$ satisfies by definition $(y_1 : \cdots : y_r) = (f_1(x) : \cdots : f_r(x))$ and hence $y_i f_j(x) = y_j f_i(x)$. But these equations then also have to hold on the closure $\tilde{X}$ of Γ_f in $X \times \mathbb{P}^{r-1}$. □

Example 3.24 (Blow-up of $\mathbb{A}^n$ at the origin). As our first example, let's consider the blow-up $\widetilde{\mathbb{A}^n}$ of affine space at $\{0\} = Z(x_1, \dots, x_n)$. We claim that in this case the inclusion of Lemma 3.23 is an equality.

Indeed, let

$$Y = \{(x, y) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : y_i x_j = y_j x_i \text{ for all } 1 \leq i, j \leq n\} \tag{1}$$

and consider the open subset $U_1 = \{(x, y) \in Y : y_1 \neq 0\}$. On U_1 we can set $y_1 = 1$ and obtain affine coordinates $x_1, \dots, x_n, y_2, \dots, y_n$. The equations for Y then say that $x_j = x_1 y_j$. So we get an isomorphisms

$$\mathbb{A}^n \to U_1, \quad (x_1, y_2, \dots, y_n) \mapsto \big((x_1, x_1y_2, \dots, x_1y_n), (1 : y_2 : \dots : y_n)\big). \tag{2}$$

We get similar morphism for the other open subsets $U_i = \{y_i \neq 0\} \subseteq Y$. Thus Y is covered by n-dimensional irreducible open subsets. But this implies that Y is n-dimensional and irreducible. It also contains the n-dimensional closed subset $\widetilde{\mathbb{A}^n}$. Thus we need to have $\widetilde{\mathbb{A}^n} = Y$.

Let's try to understand how $\mathbb{A}^n$ looks. We have a morphism $\pi : \widetilde{\mathbb{A}^n} \to \mathbb{A}^n$. By construction, π is an isomorphism on $\mathbb{A}^n - \{0\}$. The exceptional set $\pi^{-1}(0)$ is given by setting $x_1 = \dots = x_n = 0$ in (1). But the all the equations become trivial and we just get

$$\pi^{-1}(0) = \{(0, y) \in \mathbb{A}^n \times \mathbb{P}^{n-1}\} \cong \mathbb{P}^{n-1}.$$

So, passing from $\mathbb{A}^n$ to $\widetilde{\mathbb{A}^n}$ leaves $\mathbb{A}^n - \{0\}$ unchanged, but replaces the origin by a $\mathbb{P}^{n-1}$.

Naively, one could think that $\widetilde{\mathbb{A}^n}$ looks like $\mathbb{A}^n$ with $\mathbb{P}^{n-1}$ sticking out of it at the origin. Of course, this cannot be correct, because then $\widetilde{\mathbb{A}^n}$ would be irreducible. To obtain a better picture, let us compute the strict transform of a line L through the origin. By construction any point $(x, y) \in \tilde{L} \subseteq L \times \mathbb{P}^{n-1}$ with $x \neq 0$ (i.e. (x, y) outside the exceptional set) must have y equal to the projective point corresponding to $k \cdot \{x\} = L$. Hence the same also holds for the closure $\tilde{L}$ and thus the $\tilde{L} \cap \pi^{-1}(0)$ is exactly the point corresponding to L is $\mathbb{P}^{n-1}$.

Thus any two line through the origin with distinct directions will become separated in the blow-up. In other words, the exceptional set parametrizes the directions in $\mathbb{A}^n$ at 0. You should look at the picture in [G2, p. 75] or [H, p. 29]. ○

Example 3.25 (Blow-up of a plane curve). Let us now compute the blow-up of the nodal curve $X = \{x_2^2 = x_1^2 + x_1^3\} \subseteq \mathbb{A}^2$ at $(0, 0)$. The best way to do this is to consider the blow-up $\tilde{X}$ as the strict transform of X inside the blowup $\widetilde{\mathbb{A}^2}$ of $\mathbb{A}^2$ at the origin. Intuitively, that blowup should separate the two directions in which X passes through the origin, i.e. $\tilde{X}$ will intersect the exceptional set of $\widetilde{\mathbb{A}^2}$ in two separate points. Thus $\tilde{X}$ will be smooth and $\pi : \tilde{X} \to X$ a resolution of singularities of X.

Let us now actually compute this. Outside of $\mathbb{A}^n - \{0\}$ the equation for X must still be true for $\tilde{X}$ in $\widetilde{\mathbb{A}^n}$, so the equation must also hold on the closure. Thus we have the equations $x_2^2 - x_1^2 - x_1^3 = 0$ and $y_1x_2 - y_2x_1 = 0$ for $\tilde{X} \subseteq \mathbb{A}^2 \times \mathbb{P}^1$ (where the second equation comes from (1)). From the second equation we get that $y_1 = \lambda x_1$ and $y_2 = \lambda x_2$ for some $\lambda \in k^*$. Multiplying the first equation by λ^2 we get

$$0 = \lambda^2(x_2^2 - x_1^2 - x_1^3) = y_2^2 - y_1^2 - y_1^2x_1.$$

On $\pi^{-1}(0)$ we have $x_1 = x_2 = 0$ and hence $\tilde{X} \cap \pi^{-1}(0)$ is given by $y_2^2 - y_1^2 = 0$, or $(y_2 - y_1)(y_2 + y_1) = 0$. Thus the exceptional set of $\tilde{X}$ consists of the two points $(1 : 1)$ and $(1 : -1)$ of $\mathbb{P}^1$.

Let us check that $\tilde{X}$ is indeed smooth. The only possible singularities are at the exceptional points $a = ((0, 0), (1 : 1))$ and $b = ((0, 0), (1 : -1))$ of $\tilde{X} \subseteq \mathbb{A}^2 \times \mathbb{P}^1$. Both points lie in the affine open chart $U_1 = \{((x_1, x_2), (y_1 : y_2)) : y_1 \neq 0\} \cong \mathbb{A}^2$ with affine coordinates as in (2). In this chart the equation for $\tilde{X}$ becomes $y_2^2 - 1 - x_1 = 0$. The Jacobian is $(-1, 2y_2)$, which always has rank 1. Thus $\tilde{X}$ is indeed non-singular. ◯

Proposition 3.26. *Let $\tilde{X}$ be the blowup of an irreducible affine algebraic set X at a proper closed subset Z. Then every irreducible component of the exceptional set $\pi^{-1}(Z)$ has codimension 1 in $\tilde{X}$.*

Remark 3.27. If Y is a non-empty irreducible closed subset of a Noetherian space X, then the *codimension* $\operatorname{codim}_X Y$ of Y in X is the supremum of all n such that there is a chain

$$Y \subseteq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subseteq X$$

of irreducible closed subsets of X. We have $\dim Y + \operatorname{codim}_X Y = \dim X$.

Proof. Let $I(Z) = (f_1, \ldots, f_r)$. Consider again the affine open subsets $U_i \subseteq \tilde{X} \subseteq X \times \mathbb{P}^{r-1}$, where the i-th projective coordinate is non-zero. By Lemma 3.23, we have

$$U_i \subseteq \{(x, y) \in X \times \mathbb{P}^{r-1} : y_i f_j(x) = y_j f_i(x) \text{ for all } 1 \leq j \leq r\}.$$

Thus if $f_i(a) = 0$ for $a \in U_i$, then also $f_j(a) = 0$. Hence $U_i \cap \pi^{-1}(Z(f_1, \ldots, f_r))$ is given by the vanishing of the single function f_i. So the codimension can be at most one. It cannot be 0, because if f_i was identically zero on a non-empty U_i, then U_i would be contained in the exceptional set. But then also its closure $\tilde{X}$ would be contained in the exceptional set and hence $\tilde{X} - Z = \emptyset$ and $\tilde{X} = \emptyset$. □

4. SCHEMES

[Introduction: what we have done so far; motivation for looking at non-reduced algebras; motivation for looking at non-k-algebras (number theory)]

The word *ring* will always mean a commutative ring with identity.

4.1. DEFINITIONS AND EXAMPLES

Definition 4.1. Let R be a ring. Then the *spectrum* $\operatorname{Spec} R$ of R is, as a set, the set of all prime ideals in R. For $\mathfrak{p} \in \operatorname{Spec} R$ we let $k(\mathfrak{p})$ be the quotient field of the domain $R/\mathfrak{p}$. For any $S \subseteq R$ we set

$$Z(S) = V(S) = \{\mathfrak{p} \in \operatorname{Spec} R : S \subseteq \mathfrak{p}\}.$$

Remark 4.2. Any element $f \in R$ can be considered as a "function" on $\operatorname{Spec} R$ as follows: For any $\mathfrak{p} \in \operatorname{Spec} R$ denote by $f(\mathfrak{p})$ the image of f under the composition $R \to R/\mathfrak{p} \to k(\mathfrak{p})$. We call $f(\mathfrak{p})$ the *value of f at p*. Note however, that these values will be in general in different fields.

If $R = k[x_1, \dots, x_n]/I(X)$ is the coordinate ring of an affine algebraic set and $\mathfrak{p}$ is a maximal ideal (i.e. a point of X), then $k(\mathfrak{p}) = \mathfrak{p}$ and for $f \in R$ the value of f at $\mathfrak{p}$ as above is the value of f at the point corresponding to $\mathfrak{p}$ in the classical sense.

Let R be a ring and $S \subseteq R$. Then

$$\begin{aligned} Z(S) &= \{\mathfrak{p} \in \operatorname{Spec} R : S \subseteq \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} R : f \in \mathfrak{p} \text{ for all } f \in S\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} R : f(\mathfrak{p}) = 0 \text{ for all } f \in S\}, \end{aligned}$$

so the new definition of Z is really a generalization of the old one.

Lemma 4.3.

(i) If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of R, then $Z(\mathfrak{a}\mathfrak{b}) = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$.

(ii) If $\{\mathfrak{a}_i\}$ is any collection of ideals of A, then $Z(\sum_i \mathfrak{a}_i) = \bigcap_i Z(\mathfrak{a}_i)$.

(iii) If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals then $Z(\mathfrak{a}) \subseteq Z(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$.

Proof. As for affine algebraic sets. □

Definition 4.4. Let R be ring. Then we define a topology on $\operatorname{Spec} R$ by taking the subsets of the form $Z(\mathfrak{a})$ as closed subsets.

Examples 4.5. Let k be an algebraically closed field.

- $\operatorname{Spec} k$ is a point.
- $\operatorname{Spec} k[x]$, consists of the maximal ideals $(x - a)$ and the zero ideal (0).
- $\operatorname{Spec} k[x, y]$: closed points: max ideals = points of $\mathbb{A}^2$. Non-closed points: corresponding to subvarieties + generic point.
- $\operatorname{Spec} \mathbb{R}[x]$.
- $\operatorname{Spec} \mathbb{Z}$: has a point (p) for each prime p with $k(p) = \mathbb{F}_p$ and the generic point with $k(0) = \mathbb{Q}$. ○

Definition 4.6. Let R be a ring, $f \in R$ and $X = \operatorname{Spec} R$. We call $D(f) = X_f = \operatorname{Spec} R - Z(f)$ the *distinguished open subset* associated to f.

As for affine algebraic sets, the distinguished open subsets form a base of the topology of $\operatorname{Spec} R$. Indeed if $Z(\mathfrak{a}) \neq \operatorname{Spec} R$ is a closed set and $\mathfrak{p} \in \operatorname{Spec} R - Z(\mathfrak{a})$, then $\mathfrak{p} \not\supseteq \mathfrak{a}$. So there is $f \in \mathfrak{a} - \mathfrak{p}$. Then $\mathfrak{p} \in D(f)$ and $D(f) \cap Z(\mathfrak{a}) = \emptyset$.

We next need to define a sheaf of rings on $\operatorname{Spec} R$. Let $U \in \operatorname{Spec} R$ be open. Then we define $\mathcal{O}(U)$ to be the set of functions $s \colon U \to \coprod_{\mathfrak{p} \in U} R_{\mathfrak{p}}$ such that $s(\mathfrak{p}) \in R_{\mathfrak{p}}$ and s is locally a quotient of elements of R. More precisely we require that for each $\mathfrak{p} \in U$ there is a neighborhood V of $\mathfrak{p}$ in U and elements $f, g \in R$ such that for all $\mathfrak{q} \in V$ we have $f \notin q$ (i.e. $f(\mathfrak{q}) \neq 0$) and $s(\mathfrak{q}) = \frac{g}{f} \in R_{\mathfrak{q}}$.

Since the requirements on s are local, $\mathcal{O}$ is indeed a sheaf. We can add and multiply elements of $\mathcal{O}(U)$ pointwise, so $\mathcal{O}$ is a actually a sheaf of rings.

Definition 4.7. The *spectrum* of a ring R is the topological space $\operatorname{Spec} R$ together with the sheaf of rings just defined. In particular, for a ring R we define *affine n-space over R* to be $\mathbb{A}^n_R = \operatorname{Spec} R[x_1, \dots, x_n]$.

Proposition 4.8. *Let R be a ring and $(\operatorname{Spec} R, \mathcal{O})$ its spectrum.*

(i) *For any $\mathfrak{p} \in \operatorname{Spec} R$ the stalk $\mathcal{O}_{\mathfrak{p}}$ is isomorphic to the local ring $R_{\mathfrak{p}}$.*

(ii) *For any element $f \in R$, the ring $\mathcal{O}(D(f))$ is isomorphic to the localized ring R_f. In particular $\mathcal{O}(\operatorname{Spec} R) \cong R$.*

The proof of the proposition is very similar to the proof of the corresponding statement for affine algebraic sets (though slightly more involved). Please read through it in [G1, Proposition 5.1.12] or [H, Proposition II.2.2.].

Example 4.9. Regular functions on $k = k[x]/(x)$ and $k = k[x]/(x^2)$. Note that the latter is no longer just given by the value at its points. ○

If $f : X \to Y$ is continuous map between two ringed spaces, where the structure sheaves are not sheaves of functions, we do not automatically have a pullback morphism $f^* : O_Y(U) \to \mathcal{O}_X(f^{-1}(U))$. Thus we need to make this pullback maps part of the data of a morphism of ringed spaces.

Definition 4.10. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces and $f : X \to Y$ a continuous map. Define $f_*\mathcal{O}_X$ to be the ring of sheaves on Y given by $(f_*\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U))$.

A *morphism of ringed spaces* from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ consisting of a continuous map $f : X \to Y$ and a morphism of sheaves $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Explicitly $f^\#$ is given by ring homomorphisms $f^\#_U : O_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ for each open $U \subseteq Y$ which are compatible with the restriction maps. The notion of a ringed space is slightly to general for our purposes. For example, in the proposition above we showed that the stalks of $\mathcal{O}_{\operatorname{Spec} R}$ are always local rings.

Definition 4.11. A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that for each point $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring. A *morphism* of a locally ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a morphism $(f, f^\#)$ of ringed spaces such that for each point $x \in X$ the induced map on stalks $f^\#_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism of local rings. [If $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ are local rings then $\varphi : A \to B$ is a local homomorphism if $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.]

The map $f_x^\#$ is obtained in the following way. For each open neighborhood V of $f(x) \in Y$ we have a homomorphism

$$f_V^\#\colon \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V)).$$

As V ranges over the neighborhoods of $f(x)$, $f^{-1}(V)$ ranges over a subset of the neighborhoods of x. Thus we get a homomorphism on limits

$$\mathcal{O}_{Y,f(x)} = \varinjlim_{V} \mathcal{O}_Y(V) \to \varinjlim_{f^{-1}(V)} \mathcal{O}_X(f^{-1}(V)) \to \mathcal{O}_{X,x}.$$

The condition on the morphisms is needed to prove the following proposition, which we obviously want to be true.

Proposition 4.12. *Let R and S be rings. Then there is a one-to-one correspondence between morphisms* $\operatorname{Spec} R \to \operatorname{Spec} S$ *and ring homomorphisms $S \to R$.*

Proof. Set $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$. First let $\varphi\colon S \to R$ be a ring homomorphism. Then φ^{-1} induces a continuous map $f\colon Y \to X$ and a morphism of local rings

$$\varphi_{\mathfrak{p}}\colon \mathcal{O}_{Y,f(\mathfrak{p})} = S_{\varphi^{-1}(\mathfrak{p})} \to R_{\mathfrak{p}} = \mathcal{O}_{X,\mathfrak{p}}.$$

For each open set $V \subseteq Y$ we obtain a morphism of rings $f^\#\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(U))$ by pre-composing any section $s\colon V \to \coprod S_{\mathfrak{q}}$ with f and post-composing with the maps $\varphi_{\mathfrak{p}}$.

Conversely, let $(f, f^\#)\colon X \to Y$ be a morphism of locally ringed spaces. We get an induced ring homomorphism

$$\varphi = f^\#\colon S = \mathcal{O}_Y(Y) \to \mathcal{O}_X(X) = R$$

on global sections. Note that we have a commutative diagram of ring homomorphisms

$$\begin{tikzcd}
\mathcal{O}_Y(Y) = S \arrow[r, "\varphi"] \arrow[d] & \mathcal{O}_X(X) = R \arrow[d] \\
\mathcal{O}_{Y,f(\mathfrak{p})} = S_{f(\mathfrak{p})} \arrow[r, "f_{\mathfrak{p}}^\#"] & \mathcal{O}_{X,\mathfrak{p}} = S_{\mathfrak{p}}.
\end{tikzcd}$$

Since $f_{\mathfrak{p}}^\#$ is a local homomorphism it follows that $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$. Thus f coincides with the map $\operatorname{Spec} S \to \operatorname{Spec} R$ induced by φ. Now one quickly checks that $f^\#$ is also induced by φ as in the first part. $\square$

Corollary 4.13. *Let $X = \operatorname{Spec} R$ and $f \in R$. Then $(D(f), \mathcal{O}_X|_{D(f)})$ is isomorphic to the affine scheme* $\operatorname{Spec} R_f$.

Proof. Recall that $D(f) = \{\mathfrak{p} \in X : f \notin \mathfrak{p}\}$. This is also a description of the prime ideals of R_f. Thus it suffices to check that the structure sheaves coincide and it is enough to check that on the distinguished open subsets of R_f. Since $(R_f)_g = R_{fg}$ this follows the description of the sections of $\mathcal{O}$ on a distinguished open. $\square$

Definition 4.14. An *affine scheme* is a locally ringed space that is isomorphic to the spectrum of some ring. A *scheme* is a locally ringed space $(X, \mathcal{O})$ in which every point of X has an open neighborhood U such that $(U, \mathcal{O}|_U)$ is an affine scheme. A *morphism* of schemes is a morphism of locally ringed spaces. We write **Sch** for the category of schemes.

We often need a relative version.

Definition 4.15. Let S be any scheme. Then a *scheme over* S is a morphism of schemes $X \to S$. A *morphism* of schemes over S is a commutative triangle

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S. & \end{tikzcd}$$

We write $\mathbf{Sch}/S$ for the category of schemes over S. If R is a ring, we set $\mathbf{Sch}/R = \mathbf{Sch}/\operatorname{Spec} R$.

Example 4.16. Affine schemes over R are just R-algebras and R-algebra homomorphisms. ○

Theorem 4.17. *Let k be an algebraically closed field. There is a fully faithful functor t from quasi-projective algebraic sets to schemes over k. For any quasi-projective algebraic set X, its topological space is homeomorphic to the closed points of $t(X)$ and its sheaf of regular function is obtained by restricting the structure sheaf of $t(X)$ via this homeomorphism.*

Proof. Check on affines and glue. See [H, Proposition II.2.6] for details. □

4.2. SOME PROPERTIES

Definition 4.18. A scheme X is *connected* if its topological space is connected. It is *irreducible* if its topological space is irreducible.

Definition 4.19. A scheme X is *reduced* if for every open set U, the ring $\mathcal{O}_X(U)$ has no nilpotent elements. Equivalently X is reduced if and only if each the local rings $\mathcal{O}_{X,P}$ have no nilpotent elements for all P.

Example 4.20. An affine scheme $X = \operatorname{Spec} A$ is irreducible if and only if the nilradical of A is prime. It is reduced if and only if the nilradical of A is 0. Hence $\operatorname{Spec} A$ is reduced and irreducible if and only if A is an integral domain. ○

Definition 4.21. A scheme X is *integral* if for every open subset $U \subseteq X$ the ring $\mathcal{O}_X(U)$ is an integral domain.

Lemma 4.22. *Let X be a scheme and $s \in \mathcal{O}_X$. Then the set*

$$X_s = \{x \in X : s \notin \mathfrak{m}_x \subseteq (\mathcal{O}_X)_x\}$$

is an open subset of X.

Proof. By covering X with open affine sets, we can assume that X is affine. Let $X = \operatorname{Spec} A$ and pick $x = \mathfrak{p} \in X$. As $\mathfrak{m}_x$ is the localization of $\mathfrak{p}$, we have $s \in \mathfrak{m}_x \subseteq (\mathcal{O}_X)_x = A_\mathfrak{p}$ if and only if $s \in \mathfrak{p}$ (where we write s both for the section and its image in the stalk). Thus $X_s = D(s)$ is open. □

Theorem 4.23. *A scheme X is integral if and only if it is reduced and irreducible.*

Proof. Clearly an integral scheme is reduced. If X is not irreducible, then we can find two disjoint open subsets U_1 and U_2. By the sheaf condition this implies that $\mathcal{O}(U_1 \cup U_2) = \mathcal{O}(U_1) \times \mathcal{O}(U_2)$, which is not an integral domain. Thus integral also implies irreducible.

Now assume that X is reduced and irreducible. We have to show that X is integral. Let U be an open subset of X and suppose we have elements $a, b \in \mathcal{O}_X(U)$ with $ab = 0$ in $\mathcal{O}_X(U)$. By the lemma, $X_a, X_b \subseteq U$ are open sets.

Let $V \subseteq U$ be an open affine set. Since both reduced and irreducible are properties that are inherited by V, Example 4.20 implies that $\mathcal{O}_X(V)$ is an integral domain. Since $0 = (ab)|_V = a|_V b|_V$, either $a|_V = 0$ or $b|_V = 0$.

Hence for every $x \in U$ there exists an open (affine) neighborhood of x where $a = 0$ or an open neighborhood, where $b = 0$. So either $x \notin X_a$ or $x \notin X_b$. In other words $X_a \cap X_b = \emptyset$. So by irreducibility, one of the sets is empty; say X_a. But then a would lie in the intersection of all prime ideals, i.e. in the nilradical which is assumed to be just 0. Thus $a = 0$ and $\mathcal{O}_X(U)$ is integral as required. □

Definition 4.24. A scheme X is called *Noetherian* if it can be covered by finitely many open affine subsets $U_i = \operatorname{Spec} A_i$ such that all A_i are Noetherian rings.

The underlying topological space of a Noetherian scheme is Noetherian, but the converse is not necessarily true.

Definition 4.25. A scheme X over Y is called *of finite type* over Y, if there is a covering of Y be open affine subsets $V_i = \operatorname{Spec} B_i$ such that $f^{-1}V_i$ can be covered by finitely many open affines $U_{i,j} = \operatorname{Spec} A_{i,j}$, where each $A_{i,j}$ is a finitely generated B_i-algebra.

In particular a scheme over a field k is a finite type if it can be covered by finitely many open affine $U_i = \operatorname{Spec} A_i$ such that each A_i is a finitely generated (=finite type) k-algebra.

Proposition 4.26. *Let k be an algebraically closed field. Then there is a one-to-one correspondence between irreducible algebraic sets over k and integral schemes of finite type over k.*

Proposition 4.27. *Let X be any scheme and $Y = \operatorname{Spec} R$ be an affine scheme. Then*

$$\operatorname{Hom}_{\mathbf{Sch}}(X, Y) = \operatorname{Hom}_{\mathbf{Rings}}(R, \mathcal{O}_X(X)).$$

Proof. This follows from the fact that we can glue morphisms: Let $\{U_i\}$ be an open affine cover of X and let $\{U_{ijk}\}$ be an open affine cover of $U_i \cap U_j$. Then giving a morphism $X \to Y$ is the same as giving morphisms $f_i\colon U_i \to Y$ such that f_i and f_j agree on $U_i \cap U_j$, i.e. such that $f_i|_{U_{ijk}} = f_j|_{U_{ijk}}$ for all i, j, k.

The morphisms f_i and $f_i|_{U_{ijk}}$ correspond exactly to ring homomorphisms $\mathcal{O}_Y(Y) \to \mathcal{O}_X(U_i)$ and $\mathcal{O}_Y(Y) \to \mathcal{O}_X(U_{ijk})$. Hence a morphism $f\colon X \to Y$ is the same as a collection of ring homomorphisms $f_i^*\colon R \to \mathcal{O}_X(U_i)$ such that the compositions $\rho^{U_i}_{U_{ijk}} \circ f_i^*$ and $\rho^{U_j}_{U_{ijk}} \circ f_j^*\colon R \to \mathcal{O}_X(U_{ijk})$ agree for all i, j, k. But by the sheaf axiom for $\mathcal{O}_X$ this is the same a ring homomorphism $R \to \mathcal{O}_Y(Y)$. □

Remark 4.28. By the proposition, every scheme X admits a unique morphism to $\operatorname{Spec}\mathbb{Z}$, corresponding to the unique ring homomorphism $\mathbb{Z} \to \mathcal{O}_X(X)$.

4.3. FIBER PRODUCTS

Theorem 4.29. *Let $f\colon X \to S$ and $g\colon Y \to S$ be morphisms of schemes. Then there is a fiber product $X \times_S Y$.*

Proof. Suppose $S = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} C$. We will show that $X \times_S Y = \operatorname{Spec}(B \otimes_A C)$. Let Z be another scheme such that we have a commutative diagram

$$\begin{tikzcd}
 & & Y \arrow[dr, "f"] & \\
Z \arrow[urr, bend left] \arrow[drr, bend right] & X \times_S Y \arrow[ur] \arrow[dr] & & S \\
 & & X \arrow[ur, "g"'] &
\end{tikzcd}$$

We need to show that there exists a unique morphism $Z \to X \times_S Y$ fitting into the diagram. By Proposition 4.27, this is the same as showing that there is a unique homomorphism of rings $B \otimes_A C \to \mathcal{O}_Z(Z)$ fitting into the diagram

$$\begin{tikzcd}
 & & B \arrow[dl] \arrow[dll, bend right] & \\
\mathcal{O}_Z(Z) & B \otimes_A C \arrow[l, dashed] & & A \arrow[ul] \arrow[dl] \\
 & & C \arrow[ul] \arrow[ull, bend left] &
\end{tikzcd}$$

But this is just the universal property of the tensor product.

For the general case, we need to cover X, Y and S by affine schemes and construct $X \times_S Y$ locally in the affine open sets and finally glue all of them together. See for example [G1, Lemma 5.4.7] or [H, theorem II.3.3]. □

Example 4.30. Let X be a scheme and $x \in X$ a point. Then there is a natural morphism $\operatorname{Spec} k(x) \to X$, mapping the unique point of $\operatorname{Spec} k(x)$ to x and pulling back a section $\varphi \in \mathcal{O}_X(U)$ to its image under $\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to k(x)$. More generally for K a field, giving a morphism $\operatorname{Spec} K \to X$ is the same as giving a point $x \in X$ and a non-trivial field homomorphism $k(x) \hookrightarrow K$.

Now let $f: Y \to X$ be a morphism and $x \in X$ a point. Then we call

$$Y_x = Y \times_X \operatorname{Spec} k(x)$$

the *fiber* of $X \to Y$ over x.

For example set $Y = X = \mathbb{A}^1_{\mathbb{C}}$ and let $f: Y \to X$ be given by $x \mapsto x^2$. Over the point $0 \in X$ (where we identify (closed) points of X with $\mathbb{C}$) the fiber of f is $\operatorname{Spec}(\mathbb{C}[x] \otimes_{\mathbb{C}[x]} \mathbb{C})$ where the maps are given by $x \mapsto x^2$ and $x \mapsto 0$ respectively. This tensor product is equal to $\mathbb{C}[x]/(x^2)$ and hence Y_0 is a non-reduced scheme. The fibers over any non-zero $a \in X$ is given by $\operatorname{Spec} \mathbb{C}[x]/(x^2 - a)$ and hence consists of the two closed points $\pm\sqrt{a}$. [picture] (Exercise: compute the fiber over the generic point $(0) \in \operatorname{Spec} k[x]$). ○

In this way we often regard a morphism $f: Y \to X$ as a *family of schemes* over X, namely as the family $\{Y_x : x \in X\}$. We sometimes say that the fibers Y_x are *deformations* of a central fiber X_0. If the fibers X_y for $y \neq 0$ are all isomorphic, we sometimes say that X_0 is a *degeneration* of X_y.

Example 4.31. $f: \mathbb{A}^3 \to \mathbb{A}^1, f(x,y,z) = x^2 + y^2 + z^2$. The fiber over 0 (a singular cone) is a degeneration of the other fibers (smooth "cylinders"). ○

Example 4.32. $Y = \operatorname{Spec} k[x,y,t]/(ty - x^2) \to X = \operatorname{Spec} k[t]$. Y is integral and is a family of integral schemes degenerating to a non-reduced fiber at 0. ○

Example 4.33. $Y = \operatorname{Spec} k[x,y,t]/(xy - t) \to X = \operatorname{Spec} k[t]$. Y is integral and is a family of integral schemes degenerating to a reducible fiber at 0. ○

Example 4.34. $Y = \operatorname{Spec} \mathbb{Z}[x,y]/(x^2 - y^2 + 5) \to X = \operatorname{Spec} \mathbb{Z}$. ○

Given a morphism $g: X' \to X$, taking the fiber product $X' \times_X Y$ yields a morphism $f': X' \times_X Y \to X'$. This is often called a *base extension* (or a *pullback*). [picture of extension by $\mathbb{A}^1$] If X is defined over a field k and $k \hookrightarrow K$ is a field extension. Then the base extension $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$ gives "X viewed as a schemes over K" (for example we could go from $\mathbb{Q}$ (which is hard to understand) to $\mathbb{C}$).

Definition 4.35. An *open subscheme* of a scheme X is scheme U whose topological space is an open subspace of X and whose structure sheaf is isomorphic to $\mathcal{O}_X|_U$. An *open immersion* is a morphism $f: Y \to X$ which induces an isomorphism of Y with an open subscheme of X.

Given an open subset $U \subseteq X$ we can make it into an open subscheme in a unique way.

Definition 4.36. A *closed immersion* is a morphism $f\colon Y \to X$ of schemes such that f induces a homeomorphism of the topological space of Y with a closed subspace of X and the induced map $f^\#\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective. A *closed subscheme* of X is an equivalence class of closed immersions, where we say that $f\colon Y \to X$ and $f'\colon Y' \to X$ are equivalent if there exists an isomorphism $i\colon Y' \to Y$ such that $f' = f \circ i$.

Remark 4.37. The surjectivity of a morphism of sheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$ is subtle. The problem is that $\operatorname{im}\varphi$ is not necessarily a sheaf. Thus we have to say that φ is surjective if $\mathscr{G}$ is the smallest subsheaf of $\mathscr{G}$ containing $\operatorname{im}\varphi$. In particular, it is not true that necessarily $\operatorname{im}\varphi(U) = \mathscr{G}(U)$ for all open $U \subseteq X$. Alternatively, φ is surjective if the induced map $\varphi_x\colon \mathscr{F}_x \to \mathscr{G}_x$ on stalks is surjective for all x.

Lemma 4.38. *Let $X = \operatorname{Spec} A$ be an affine scheme and $\mathfrak{a} \subseteq A$ an ideal. Then the ring homomorphism $A \to A/\mathfrak{a}$ induces a closed immersion of $\operatorname{Spec} A/\mathfrak{a}$ onto $Z(\mathfrak{a}) \subseteq X$.*

Proof. Clearly $\operatorname{Spec} A/\mathfrak{a} \to Z(\mathfrak{a})$ is a homeomorphism onto a closed subset of X. The map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, because it is surjective on stalks, which are localizations of A and $A/\mathfrak{a}$ respectively. □

Thus if $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$, we get two different structures of closed subscheme on $Z(\mathfrak{a}) = Z(\mathfrak{b})$.

Let $i_1\colon Z_1 \hookrightarrow X$ and $i_2\colon Z_2 \hookrightarrow X$ be closed immersions. Then we define the *scheme-theoretic* intersection of Z_1 and Z_2 in X to be the fiber product $Z_1 \cap Z_2 = Z_1 \times_X Z_2$, realized as a subscheme of X via either projection map. (We won't show now that this actually makes sense in general.)

If $X = \operatorname{Spec} A$, Z_1 is given by $\mathfrak{a}$ and Z_2 by $\mathfrak{b}$, then

$$Z_1 \times_X Z_2 = \operatorname{Spec}(A/\mathfrak{a} \otimes_A A/\mathfrak{b}) = \operatorname{Spec}(A/(\mathfrak{a} + \mathfrak{b})).$$

Example 4.39. Parabola intersected with line (again…) ○

Definition 4.40. Let $f\colon X \to Y$ be a morphism of schemes. The *diagonal morphism* is the unique map $\Delta\colon X \to X \times_Y X$ whose composition with both projection maps is the identity.

$$\begin{tikzcd}
 & & X \arrow[dr, "f"] & \\
X \arrow[r, dashed, "\Delta"] \arrow[urr, bend left, "\mathrm{Id}"] \arrow[drr, bend right, "\mathrm{Id}"'] & X \times_Y X \arrow[ur] \arrow[dr] & & Y \\
 & & X \arrow[ur, "f"'] &
\end{tikzcd}$$

The morphism f is called *separated* if Δ is a closed immersion. In this case we sometimes say that X is separated over Y. In particular, we say that a scheme X is *separated* if it is separated over $\operatorname{Spec}\mathbb{Z}$.

Theorem 4.41. *Every morphism of of affine schemes is separated.*

Proof. Let $f: X = \operatorname{Spec} A \to \operatorname{Spec} B = Y$ be a morphism of affine schemes, given by a homomorphism $B \to A$. Then $X \times_Y X = \operatorname{Spec}(A \otimes_B A)$ and the diagonal morphism corresponds to the homomorphism $A \otimes_B A \to A$ given by $a \otimes a' \mapsto aa'$. This is surjective, so by Lemma 4.38 the diagonal morphism Δ is a closed immersion. □

Theorem 4.42. *An arbitrary morphism $f: X \to Y$ is separated if and only if the image of the diagonal morphism is a closed subset of $X \times_Y X$.*

Proof. The "only if"-direction is true by definition. So assume that $\Delta(X)$ is a closed subset of X. We have to show that Δ is actually a closed immersion. Consider the first projection $p: X \times_Y X \to X$. Then $p \circ \Delta = \mathrm{Id}_X$; hence Δ is a homeomorphism onto its image, which is closed. It remains to be shown that $\mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X$ is surjective. This is a local question. For $Q \notin \Delta(X)$, the induced map on the stalk at Q is certainly surjective. So let $P \in X$ and choose an open affine neighborhood U of P in X such that $f(U)$ is contained in an open affine subset of V of Y. Then $U \times_V U$ is an open affine neighborhood of $\Delta(P)$. By Theorem 4.41 above, $\Delta: U \to U \times_V U$ is separated. Therefore the map of sheaves is surjective in a neighborhood of $\Delta(P)$. Since P was arbitrary, $\mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X$ is surjective as required. □

Theorem 4.43. *The following morphisms of Noetherian schemes are separated.*

- *Open and closed immersions;*
- *Compositions of separated morphisms;*
- *Base extensions of separated morphisms (by any morphism).*

Proof. [H, Corollary II.4.6] □

Definition 4.44. A morphism is called *proper* if it is separated, of finite type and universally closed.

Theorem 4.45. *The following morphisms of Noetherian schemes are proper.*

- *Closed immersions;*
- *Compositions of proper morphisms;*
- *Base extensions of proper morphisms (by any morphism).*

Proof. [H, Corollary II.4.8] □

The main example of proper morphisms will be projective morphisms.

4.4. PROJECTIVE SPACE

By a *graded ring* we will mean a (commutative, unital) ring R together with a decomposition $R = \bigoplus_{d\geq 0} R_d$ into abelian groups such that $R_d \cdot R_e \subseteq R_{d+e}$. An element of R_d is called *homogeneous of degree d*. An ideal $\mathfrak{a} \subseteq R$ is called *homogeneous* if it can be generated by homogeneous elements. We let R_+ be the ideal $\bigoplus_{d>0} R_d$.

Definition 4.46. We define $\operatorname{Proj} R$ to be the set of all homogeneous prime ideals $\mathfrak{p} \subseteq R$ with $R_+ \nsubseteq \mathfrak{p}$. For a homogeneous ideal $\mathfrak{a}$ we define $Z(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj} R : \mathfrak{p} \supseteq \mathfrak{a}\}$.

As usual, one can prove the following lemma and define a topology on $\operatorname{Proj} R$.

Lemma 4.47. *Let R be a graded ring.*

(i) *If $\{\mathfrak{a}_i\}$ is any family of homogeneous ideals of R, then $\bigcap_i Z(\mathfrak{a}_i) = Z(\sum_i \mathfrak{a}_i)$.*

(ii) *If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ a homogeneous ideals of R, then $Z(\mathfrak{a}_1) \cup Z(\mathfrak{a}_2) = Z(\mathfrak{a}_1 a_2)$.*

Definition 4.48. Let R be a graded ring and $\mathfrak{p}$ a homogeneous prime ideal. We set

$$R_{(\mathfrak{p})} = \left\{\frac{g}{f} : g \notin \mathfrak{p} \text{ and } f, g \in R_d \text{ for some } d\right\}$$

to be the ring of degree zero elements of the localization of R with respect to the multiplicative system of all homogeneous elements in R that are not in $\mathfrak{p}$.

For any open subset $U \subseteq \operatorname{Proj} R = X$ we define $\mathcal{O}_X(U)$ to be the set of all functions $s\colon U \to \coprod_{\mathfrak{p}\in U} R_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U$, $s(p) \in R_{(\mathfrak{p})}$ and such that s is locally a quotient of elements of R: for each $\mathfrak{p} \in U$ there exists an open neighborhood V on $\mathfrak{p}$ in U and homogeneous elements $g, f \in R$ of the same degree, such that for all $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{g}{f} \in R_{(\mathfrak{q})}$.

One the proves the usual proposition whose proof can be found in [H, Proposition II.2.5] or [G1, Proposition 5.5.4].

Proposition 4.49. *Let R be a graded ring and set $X = \operatorname{Proj} R$ with the structure sheaf just defined.*

(i) *For every $\mathfrak{p} \in \operatorname{Proj} R$ the stack $\mathcal{O}_{X,\mathfrak{p}}$ is isomorphic to $R_{(\mathfrak{p})}$.*

(ii) *For any homogeneous elements $f \in R_+$, let $D_+(f) = X_f \subseteq X$ be the distinguished open subset*

$$X_f = X - Z(f) = \{\mathfrak{p} \in \operatorname{Proj} R : f \notin \mathfrak{p}\}.$$

These open sets cover X and for each such open subset we have an isomorphism of locally ringed spaces $(X_f, \mathcal{O}_X|_{X_f}) \cong \operatorname{Spec} R_{(f)}$, where

$$R_{(f)} = \left\{\frac{g}{f^r} : g \in R_{r\cdot \deg f}\right\}$$

is the ring of elements of degree zero in the localization R_f.

In particular, Proj R *is a scheme.*

Definition 4.50. Let A be a ring. We define *projective n-space over A* to be the scheme

$$\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n],$$

where $\deg x_i = 1$.

In particular, if k is an algebraically closed field, then $\mathbb{P}^n_k$ is a scheme whose closed points are homeomorphic to the *algebraic set* called projective n-space.

The first part of the following lemma is an analogue of Lemma 2.34.

Lemma 4.51. *Let R and S be graded rings.*

(i) Let $\varphi : R \to S$ be a graded homomorphism of graded rings (preserving degree). Let $U = \{\mathfrak{p} \in \operatorname{Proj} S : \mathfrak{p} \not\supseteq \varphi(R_+)\}$. Then U is an open subset of Proj S *and φ determines a natural morphism $f : U \to \operatorname{Proj} R$.*

(ii) Let φ be in addition surjective. Then $U = \operatorname{Proj} S$ and $f : \operatorname{Proj} S \to \operatorname{Proj} R$ is a closed immersion.

Proof. Exercise [H, Exercises 2.14 and 3.12]. □

In particular, if $\mathfrak{a}$ is a homogeneous ideal of R, then we get a closed immersion $\operatorname{Proj} R/\mathfrak{a} \hookrightarrow \operatorname{Proj} R$. We remark that the existence of the "irrelevent ideal" implies that different homogeneous ideals can lead to the same closed subscheme. For example, the ideals (f) and $(x_0 f, x_1 f, \dots, x_n f)$ define the same closed subscheme.

Remark 4.52. If $X \subseteq \mathbb{P}^n$ is a projective algebraic set, then its associated scheme is $\operatorname{Proj} S(X) \hookrightarrow \mathbb{P}^n_k$, where $S(X) = k[x_0, \dots, x_n]/\tilde{I}(X)$.

5. BÉZOUT'S THEOREM

We will fix an algebraically closed ground field k, and only consider schemes over k. In particular we are interested in closed subschemes of projective space $\mathbb{P}^n = \mathbb{P}^n_k$.

As we stated before, each homogeneous ideal $\mathfrak{a} \subseteq k[x_0, \dots, x_n]$ yields a closed subscheme of $\mathbb{P}^n$. In fact the reverse is also true [H, Corollary II.5.16], though we won't prove this here. Given a closed subscheme X of $\mathbb{P}^n$ defined by a homogeneous ideal $\mathfrak{a}$ we will write $S(X) = k[x_0, \dots, x_n]/\mathfrak{a}$ (really, we should use the *saturation* of $\mathfrak{a}$ [G1, Definition 5.5.7] to make this unique, but it doesn't really matter for our purposes). Clearly $S(X)$ is a graded ring, and we write $S(X)_d$ for the degree d part.

5.1. THE HILBERT POLYNOMIAL

Our first goal is to define a some kind of "intersection multiplicity". [motivational example]

Definition 5.1. Let X be a closed subscheme of $\mathbb{P}^n$. The *Hilbert function* of X is the function

$$h_X \colon \mathbb{Z} \to \mathbb{Z}, \quad h_X(d) = \dim_k S(X)_d.$$

Example 5.2. If $X = \mathbb{P}^n$, then $S(X) = k[x_0, \dots, x_n]$ and $\dim_k S(X)_d = \binom{d+n}{n}$. Thus

$$h_X(d) = \binom{d+n}{n} = \frac{(d+n)(d+n-1)\cdots(d+1)}{n!}$$

is actually a polynomial in d of degree n with leading coefficient $\frac{1}{n!}$. ○

Example 5.3. Let us now consider some zero-dimensional subschemes (i.e. collections of points).

(i) Let $X = \{(1:0), (0:1)\} \subseteq \mathbb{P}^1$ be two points. Then $S(X) = k[x_0, x_1]/(x_0x_1)$ and $S(X)_d = \{x_0^d, x_1^d\}$ for $d \geq 1$. Thus

$$h_X(d) = \begin{cases} 1 & \text{if } d = 0, \\ 2 & \text{if } d \geq 1. \end{cases}$$

(ii) Let $X = \{(1:0:0), (0:1:0), (0:0:1)\} \subseteq \mathbb{P}^2$ be three non-colinear points. Then $S(X) = k[x_0, x_1, x_2]/(x_0x_1, x_1x_2, x_0x_2)$ and $S(X)_d = \{x_0^d, x_1^d, x_2^d\}$ for $d \geq 1$. Thus

$$h_X(d) = \begin{cases} 1 & \text{if } d = 0, \\ 3 & \text{if } d \geq 1. \end{cases}$$

(iii) Let $X = \{(1:0), (0:1), (1:1)\} \subseteq \mathbb{P}^1$ be three *colinear* points. Then $S(X) = k[x_0, x_1]/(x_0x_1(x_0 - x_1))$. Now we have $S(X)_1 = \{x_0, x_1\}$ and $S(X)_d = \{x_0^d, x_0x_1^{d-1}, x_1^d\}$ for $d \geq 1$. Thus

$$h_X(d) = \begin{cases} 1 & \text{if } d = 0, \\ 2 & \text{if } d = 1, \\ 3 & \text{if } d \geq 2. \end{cases}$$

(iv) Let $X \subseteq \mathbb{P}^1$ be the "double point" given by $\mathfrak{a} = (x_0^2)$. Then $S(X)_d = \{x_0x_1^{d-1}, x_1^d\}$ and

$$h_X(d) = \begin{cases} 1 & \text{if } d = 0, \\ 2 & \text{if } d \geq 1. \end{cases}$$

We see that in all these cases, while $h_X(d)$ can vary for small d, eventually it counts the points of X with the expected multiplicity. ○

Lemma 5.4. *Let X be a zero-dimensional closed subscheme of $\mathbb{P}^n$. Then:*

(i) X is affine.

(ii) If we write $X = \operatorname{Spec} R$ for some k-algebra R, then R is a finite dimensional vector space over k.

(iii) $h_X(d) = \dim_k R$ for $d \gg 0$. In particular $h_X(d)$ is constant for large values of d.

Definition 5.5. The dimension of R in the Proposition is called the *length* of X. We interpret it as the number of points in X, counted with their scheme-theoretic multiplicities. It follows from the proof of part (ii), that for a reduced scheme it is exactly the number of points.

Proof of Lemma 5.4. (i) Since X is zero-dimensional, we can find a hyperplane H of $\mathbb{P}^n$ that does not intersect X. Then X is a closed subscheme of the affine scheme $X - H$.

(ii) If X is reducible, say $X = X_1 \sqcup \cdots \sqcup X_m$ with $X_i = \operatorname{Spec} R_i$, then $R = R_1 \times \cdots \times R_n$ (exercise). Thus it suffices to show that each of the R_i is finite dimensional and we can assume that X is irreducible. Further we can apply a linear change of coordinates and assume that X is the origin in $\mathbb{A}^n$.

Let $X = \operatorname{Spec} k[x_1, \dots, x_n]/\mathfrak{b}$. Then since X is the origin, we must have $\sqrt{\mathfrak{b}} = (x_1, \dots, x_n)$ by the Nullstellensatz. Hence $x_i^d \in \mathfrak{b}$ for all i and some sufficiently large r. So, by the pigeonhole principle, every monomial of degree at least $D = rn$ lies in $\mathfrak{a}$. In other words, $R = k[x_1, \dots, x_n]/\mathfrak{b}$ has a basis consisting of monomial of degree less than D. Thus R is finite dimensional.

(iii) Let X be defined by $\mathfrak{a} \subseteq k[x_0, \dots, x_n]$. Then $\mathfrak{b} = \mathfrak{a}|_{x_0=1}$, and conversely $\mathfrak{a}$ is the homogenization of $\mathfrak{b}$. So for D as before and any $d \geq D$ we have an isomorphism of vector spaces $S_d \to R$

$$(k[x_0, \dots, x_n]/\mathfrak{a})_d \to k[x_1, \dots, x_n]/\mathfrak{b}, \quad f \mapsto f|_{x_0=1}$$

with inverse

$$k[x_1, \dots, x_n]/\mathfrak{b} \to (k[x_0, \dots, x_n]/\mathfrak{a})_d, \quad f \mapsto f^h \cdot x_0^{d-\deg f}.$$

We note that the second map is well-defined, since $k[x_1, \dots, x_n]/\mathfrak{b}$ has a basis consisting of polynomials of degree less than D.

□

We next want to prove a generalization of this result to higher dimensions. The proof will be inductive and to do the induction step we first need another lemma (Gathmann completely ignores this…).

Lemma 5.6. *Let X and Y be closed subschemes of $\mathbb{P}^n$ or $\mathbb{A}^n$ of dimensions r and s respectively. Then $X \cap Y$ has dimension at least $r + s - n$.*

Proof. By decomposing into irreducible components, we can assume that X and Y are irreducible. Further, by covering $\mathbb{P}^n$ with the standard affine open subsets, it suffices to prove the theorem for subschemes of $\mathbb{A}^n$.

Let $X = \operatorname{Spec} k[x_1, \dots, x_n]/\mathfrak{a}$ and $Y = \operatorname{Spec} k[x_1, \dots, x_n]/\mathfrak{b}$. Then the scheme-theoretic intersection $X \cap Y$ is $\operatorname{Spec} k[x_1, \dots, x_n]/(\mathfrak{a} + \mathfrak{b})$. But this has the same dimension as the *set-theoretic* intersection $Z(\sqrt{\mathfrak{a} + \mathfrak{b}})$ of the *algebraic sets* $X_{\text{red}} = \operatorname{Spec} k[x_1, \dots, x_n]/\sqrt{\mathfrak{a}}$ and $Y_{\text{red}} = \operatorname{Spec} k[x_1, \dots, x_n]/\sqrt{\mathfrak{b}}$. Thus we can prove this theorem in the setting of irreducible affine algebraic sets.

Now first suppose that Y is a hypersurface, given by an equation $f = 0$. If $X \subseteq Y$, then there is nothing to prove. Otherwise, the irreducible components of $X \cap Y$ correspond to the minimal prime ideals over the principal ideal (f) in $A(X) = k[x_1, \dots, x_n]/\sqrt{\mathfrak{a}}$. By the Krull principal ideal theorem, each such minimal prime $\mathfrak{p}$ has height one, so that $A(X)/\mathfrak{p}$ has dimension $r - 1$. Thus each irreducible component of $X \cap Y$ has dimension $r - 1$.

For the general case, consider the product $X \times Y$ in $\mathbb{A}^{2n}$, which is an irreducible affine algebraic set of dimension $r + s$. Let Δ be the diagonal of $\mathbb{A}^{2n}$. Then under the isomorphism $\mathbb{A}^n \to \Delta$, $P \mapsto (P, P)$, the subset $X \cap Y$ corresponds to $(X \times Y) \cap \Delta$. Since Δ has dimension n and $(r+s)+n-2n = r+s-n$, we are reduced to proving the result for the two irreducible affine algebraic sets $X \cap Y$ and Δ of $\mathbb{A}^{2n}$. But now Δ is the intersection of exactly n hypersurfaces, namely $x_1 - y_1 = 0, \dots, x_n - y_n = 0$. Thus applying the case of a hypersurface n times yields the result. $\square$

Proposition 5.7. *Let X be a (non-empty) m-dimensional closed subscheme of $\mathbb{P}^n$. Then there is a unique polynomial $\chi_X \in \mathbb{Q}[d]$ such that $\chi_X(d) = h_X(d)$ for $d \gg 0$. Moreover,*

(i) The degree of χ_X is m.

(ii) The leading coefficient is $\frac{1}{m!}$ times a positive integer.

Definition 5.8. The polynomial χ_X is called the *Hilbert polynomial* of X (in $\mathbb{P}^n$). The *degree* $\deg X$ of X is defined to be $(\dim X)!$ times the leading coefficient of χ_X. By the proposition this is a positive integer.

Example 5.9. (i) If X is zero-dimensional, then $\deg X$ is just the length of X, i.e. the "number of points of X with multiplicities".

(ii) $h_{\mathbb{P}^n}(d) = \chi_{\mathbb{P}^n}(d) = \frac{(d+n)(d+n-1)\cdots(d+1)}{n!}$. Thus $\deg \mathbb{P}^n = 1$.

(iii) Let $X = \operatorname{Proj} k[x_0, \dots, x_n]/(f)$ be the zero locus of a homogeneous polynomial. Then $\deg X = \deg f$. Indeed, for $d \geq \deg f$, looking at the d-th graded part of $S(X) =$

$k[x_0,\dots,x_n]/f\cdot k[x_0,\dots,x_n]$ we get

$$\begin{aligned}
h_X(d) &= \dim_k k[x_0,\dots,x_n]_d - \dim_k k[x_0,\dots,x_n]_{d-\deg f}\\
&= \binom{d+n}{n} - \binom{d-\deg f+n}{n}\\
&= \frac{1}{n!}\big((d+n)\cdots(d+1)-(d-\deg f+n)\cdots(d-\deg f-1)\big)\\
&= \frac{\deg f}{(n-1)!}d^{n-1} + \text{lower order terms.}
\end{aligned}$$

○

Proof of Proposition 5.7. We will proceed by induction on the dimension m of X. The base case $m=0$ is Lemma 5.4. So assume that $m\geq 1$. By a linear change of coordinates we can assume that no component of X lies in the hyperplane $H=\{x_0=0\}$. Write $X=\operatorname{Proj} k[x_0,\dots,x_n]/\mathfrak{a}$. We claim that there is an exact sequence of graded vector spaces over k

$$0\to k[x_0,\dots,x_n]/\mathfrak{a}\xrightarrow{\cdot x_0} k[x_0,\dots,x_n]/\mathfrak{a}\to k[x_0,\dots,x_n]/(\mathfrak{a}+(x_0))\to 0. \tag{3}$$

Indeed the only non-trivial assertion is that the first arrow is injective. So assume that it is not injective. Then there exists a polynomial $f\in k[x_0,\dots,x_n]$ such that $f\notin\mathfrak{a}$, but $x_0f\in\mathfrak{a}$. The last conditions says that f vanishes on all of X except possibly on $X\cap H$. Thus $X=(X\cap Z(f))\cup(X\cap H)$. Since $f\notin\mathfrak{a}$ we must have $X\cap Z(f)\neq X$, and then there must be an irreducible component of X contained in $X\cap H$, a contradiction to our initial assumption.

Taking d-the graded parts in (3), we deduce that

$$h_{X\cap H}(d) = h_X(d) - h_X(d-1).$$

By Lemma 5.6, $\dim X\cap H = m-1$, so by the induction assumption, $h_{X\cap H}(d)$ is a polynomial of degree $m-1$ for large d with leading coefficient $\frac{1}{(m-1)!}$.

We can write

$$h_{X\cap H}(d) = \sum_{i=0}^{m-1} c_i\binom{d}{i} \qquad \text{for } d\gg 0,$$

for some constants $c_i\in\mathbb{Q}$, where c_{m-1} is a positive integer (note that $\binom{d}{i}$ is a polynomial of degree i in d with leading coefficient $\frac{1}{i!}$). Set

$$P(d) = \sum_{i=0}^{m-1} c_i\binom{d+1}{i+1}.$$

Then P has degree m with leading coefficient $\frac{1}{m!}$ times a positive integer. Then

$$P(d)-P(d-1) = \sum_{i=0}^{m-1} c_i\binom{d+1}{i+1} - \sum_{i=0}^{m-1} c_i\binom{d}{i+1} = \sum_{i=0}^{m-1} c_i\binom{d}{i} = h_{X\cap H}(d) = h_X(d)-h_X(d-1)$$

for sufficiently large d. Thus $(P - h_X)(d)$ fulfills the difference equation $\Delta(P - h_X)(d) = 0$ for $d \gg 0$. Hence there exists an integer c such that $h_X(d) = P(d) + c$ for $d \gg 0$ by induction. □

Let us finish this section with a useful observation for computing degrees.

Proposition 5.10. *Let X_1 and X_2 be m-dimensional projective subschemes of $\mathbb{P}^n$ and assume that $\dim(X_1 \cap X_2) < m$. Then $\deg(X_1 \cup X_2) = \deg X_1 + \deg X_2$.*

Proof. Let $X_i = k[x_0, \dots, x_n]/\mathfrak{a}_i$. Then $X_1 \cap X_2 = k[x_0, \dots, x_n]/(\mathfrak{a}_1 + \mathfrak{a}_2)$ and $X_1 \cup X_2 = k[x_0, \dots, x_n]/(\mathfrak{a}_1 \cap \mathfrak{a}_2)$. So from the exact sequence

$$0 \to k[x_0, \dots, x_n]/(\mathfrak{a}_1 \cap \mathfrak{a}_2) \xrightarrow{f \mapsto (f,f)} k[x_0, \dots, x_n]/\mathfrak{a}_1 \oplus k[x_0, \dots, x_n]/\mathfrak{a}_2 \xrightarrow{(f,g) \mapsto f-g} k[x_0, \dots, x_n]/(\mathfrak{a}_1 + \mathfrak{a}_2) \to 0$$

we conclude that

$$h_{X_1}(d) + h_{X_2}(d) = h_{X_1 \cup X_2}(d) + h_{X_1 \cap X_2}(d).$$

Thus the same equation is true for the Hilbert polynomial. Since the degree of $\chi_{X_1 \cap X_2}$ is smaller than m, we obtain the statement from comparing the leading (i.e. m-th) coefficients. □

5.2. Bézout's theorem and some corollaries

Theorem 5.11. *Let X be a closed subscheme of $\mathbb{P}^n$ of positive dimension and let $f \in k[x_0, \dots, x_n]$ be a homogeneous polynomial such that no component of X is contained in $Z(f)$. Then*

$$\deg(X \cap Z(f)) = \deg X \cdot \deg f.$$

Proof. As in the proof of Proposition 5.7 we have an exact sequence

$$0 \to k[x_0, \dots, x_n]/\mathfrak{a} \xrightarrow{\cdot x_0} k[x_0, \dots, x_n]/\mathfrak{a} \to k[x_0, \dots, x_n]/(\mathfrak{a} + (x_0)) \to 0. \tag{4}$$

Hence,

$$\chi_{X \cap Z(f)}(d) = \chi_X(d) - \chi_X(d - \deg f).$$

Let $m = \dim X$ and write

$$\chi_X(d) = \frac{\deg X}{m!} d^m + c_{m-1} d^{m-1} + O(d^{m-2}).$$

Thus

$$\begin{aligned} \chi_{X \cap Z(f)} &= \frac{\deg X}{m!}(d^m - (d - \deg f)^m) + c_{m-1}(d^{m-1} - (d - \deg f)^m) + O(d^{m-2}) \\ &= \frac{\deg X}{m!} m \deg f \cdot d^{m-1} + O(d^{m-2}). \end{aligned}$$

Thus $\deg X \cap Z(f) = \deg X \deg f$. □

Corollary 5.12 (Bézout's theorem)**.** *Let X_1, …, X_n be hypersurfaces in $\mathbb{P}^n$ given by homogeneous polynomials of degree d_1, …, d_n. If the X_i have no components in common, then their intersection is zero-dimensional and*

$$\deg(X_1 \cap \cdots \cap X_n) = d_1 \cdots d_n.$$

Proof. Induction on Theorem 5.11. □

Specializing to $n = 2$ we finally obtain the original version of Bézout's Theorem.

Corollary 5.13 (Bézout's theorem, planar version)**.** *Let C_1 and C_2 be two curves in $\mathbb{P}^2$. Then*

$$\deg(C_1 \cap C_2) = \deg C_1 \cdot \deg C_2.$$

In this setting if P is a point in the (set-theoretic) intersection $\deg(C_1 \cap C_2)$, one often calls the summand in $\deg(C_1 \cap C_2)$ corresponding to P the *intersection multiplicity* of C_1 and C_2 at P. Explicitly, let C_i is given by $f_i = 0$ and choose affine coordinates around P. Write $P = (a, b)$ and let $\tilde{f}_i$ be the dehomogenizations of f_i. Then we have

$$i(C_1, C_2; P) = \dim_k \big(k[x_1, x_2]/(\tilde{f}_1, \tilde{f}_2)\big)_{(x-a, y-b)}.$$

Bézout's theorem then reads

$$\sum_{P \in C_1 \cap C_2} i(C_1, C_2; P) = \deg f_1 \cdot \deg f_2.$$

The intersection multiplicities have geometric meaning as follows.

Lemma 5.14. *Let C_1 and C_2 be two curves in $\mathbb{P}^2$ intersecting at a (closed) point P. Then:*

(i) If C_1 and C_2 are smooth at P and have different tangent lines at P, then $i(C_1, C_2; P) = 1$.

(ii) If C_1 and C_2 are smooth at P and are tangent to each other at P, then $i(C_1, C_2; P) \geq 2$.

(iii) If either C_1 or C_2 are singular at P, then $i(C_1, C_2; P) \geq 2$.

(iv) If both C_1 and C_2 are singular at P, then $i(C_1, C_2; P) \geq 3$.

Proof. Since the intersection multiplicity is local at P, we can assume that the curves are affine in $\mathbb{A}^2$ and $P = (0, 0)$. Let

$$C_i = \{f_i = 0\}, \qquad \text{where } f_i = a_i x + b_i y + \text{higher order terms}.$$

Note that C_i is singular at the origin if and only if $a_i = b_i = 0$ and the curves are tangent if and only if the vectors (a_1, b_1) and (a_2, b_2) are linearly dependent. This gives information about the linearly independent elements of $k[x, y]/(f_1, f_2)$ For example if both curves are singular (i.e. all linear coefficients vanish), then $k[x, y]/(f_1, f_2)$ contains at least the linearly independent elements 1, x and y. Thus $i(C_1, C_2; P) \geq 3$ in this case. The other statements are proven in a similar manner. □

Remark 5.15. Classically the degree of a dim r subvariety of $\mathbb{P}^n$ is the number of points of intersection (counted *without* multiplicity) with a generic linear subspace of dimension $n - r$.

Bézout's theorem (and its generalizations) are very useful tools.

Corollary 5.16 (Pascal's theorem)**.** *Let $X \subseteq \mathbb{P}^2$ be a conic (i.e. given by a homogeneous polynomial of degree* 2*). Pick six distinct points A, B, C, D, E and F on X. Then the points $P = \overline{AB} \cap \overline{DE}$, $Q = \overline{CD} \cap \overline{FA}$ and $R = \overline{EF} \cap \overline{BC}$ lie on a line.*

Remark 5.17. In this chapter we assume that the base field k is algebraically closed. However, the real version of the theorem follows from the complex version: If X and the points are defined over $\mathbb{R}$, then so are P, Q and R and hence also the line through them.

Proof. If X is the union of two lines, then the statement is trivial. Hence we can assume that X is irreducible.

Consider the two irreducible cubics $X_1 = \overline{AB} \cup \overline{CD} \cup \overline{EF}$ and $X_2 = \overline{BC} \cup \overline{DE} \cup \overline{FA}$. Let $f_i = 0$ be the (homogeneous) equation for X_i. By Bézout's theorem the two cubics X_1 and X_2 meet in exactly the $3 \cdot 3$ points A, B, C, D, E, F, P, Q and R.

Pick any point $S \in X$ not equal to the six points already chosen. Then there exists a linear combination $\lambda f_1 + \mu f_2$ that vanishes at S. Consider the cubic curve $X' = Z(\lambda f_1 + \mu f_2)$.

The curves X and X' meet at least in the 7 points A, B, C, D, E, F and S. But $\deg X' \cdot \deg X = 6$. So by Bézout's theorem X and X' must have a common component. Since we assume that X is irreducible, we thus must have that X' is reducible and $X' = X \cup L$ for some line L.

Finally, the points P, Q and R must lie on X' since they lie both on X_1 and X_2. But they are not all on X (since X is irreducible and hence can have at most two points of intersection with any line), so they must be on L. □

Corollary 5.18. *Let C be an irreducible plane curve of degree d. Then C has at most $\binom{d-1}{2}$ singular points.*

Proof. For $d = 1$, C is a line and hence smooth. Similarly, every irreducible conic is smooth. Hence we can assume that $d \geq 3$.

Assume that the statement was false and we have $\binom{d-1}{2} + 1$ distinct singular points $P_1, \dots, P_{\binom{d-1}{2}+1}$ of C. Pick additional distinct points $Q_1, \dots, Q_{d-3}$ of C. Thus in total we have $\frac{d^2}{2} - \frac{d}{2} - 1$ points.

There is a curve C' of degree $d - 2$ passing through all these points. To see this, note that the space of homogeneous polynomial of degree $(d - 2)$ in three variables has dimension $\binom{d}{2}$. In other words, the space of curves of degree $d - 2$ in $\mathbb{P}^2$ is the projective space $\mathbb{P}^N$ with $N = \binom{d}{2} - 1$, with the coefficients of the equation as homogeneous coordinates. The condition, that the curve passes through a given point is a linear condition in this $\mathbb{P}^N$. We have exactly $N = \binom{d}{2} - 1 = \frac{d^2}{2} - \frac{d}{2} - 1$ such conditions. But N hyperplanes in $\mathbb{P}^N$ always have a common point of intersection, so there is curve of degree $d - 2$ passing through the N given points.

Now consider the intersection $C \cap C'$. It contains the $d - 3$ points Q_i and the $\binom{d-1}{2} + 1$ singular points P_j. Assuming that C and C' have no common component, the latter points count with multiplicity at least two by Lemma 5.14. Thus,

$$\deg(C \cap C') \geq (d-3) + 2\left(\binom{d-1}{2} + 1\right) = d^2 - 2d + 1.$$

But $\deg C \cdot \deg C' = d(d-2) = d^2 - 2d$, a contradiction to Bézout's theorem. Thus the two curves must have a common component. But C is irreducible of degree $\deg C > \deg C'$, so this is impossible. Thus our original assumption of having more than $\binom{d-1}{2}$ singular points must be false. □

6. THE FUNCTOR OF POINTS

We will discuss different type of points. To avoid confusion, we will now on write $X = (X^{\text{top}}, \mathcal{O}_X)$ for the underlying topological space and the structure sheaf.

6.1. MOTIVATION

Motivation: Spec $\mathbb{Z}[x]$

(i) (0);

(ii) (p), for $p \in \mathbb{Z}$ prime;

(iii) principal ideals of the form (f), where $f \in \mathbb{Z}[x]$ is a polynomial irreducible over $\mathbb{Q}$ whose coefficients have greatest common divisor 1; and

(iv) maximal ideals of the form (p, f), where $p \in \mathbb{Z}$ is a prime and $f \in \mathbb{Z}[x]$ a monic polynomial whose reduction mod p is irreducible.

Motivation: Zariski topology of products
We want to obtain a way to understand "points" of schemes in a better way.
Motivation: Other categories (groups, top)
More generally, given a category $\mathbf{C}$ and a fixed object $z \in \mathbf{C}$, we can look at the functor

$$X \mapsto \mathrm{Hom}_{\mathbf{C}}(z, X).$$

For a well chosen object z we might be able to identify the set $\mathrm{Hom}_{\mathbf{C}}(z, X)$ with the points of X. In particular if X wasn't given by a point-set to begin with, we might obtain one this way.

Of course we want this functor to be faithful, i.e. given a map $f : X_1 \to X_2$ in $\mathbf{C}$, we want that the induced map

$$\mathrm{Hom}_{\mathbf{C}}(z, X_1) \to \mathrm{Hom}_{\mathbf{C}}(z, X_2)$$

determines f. The question then becomes how to choose z.

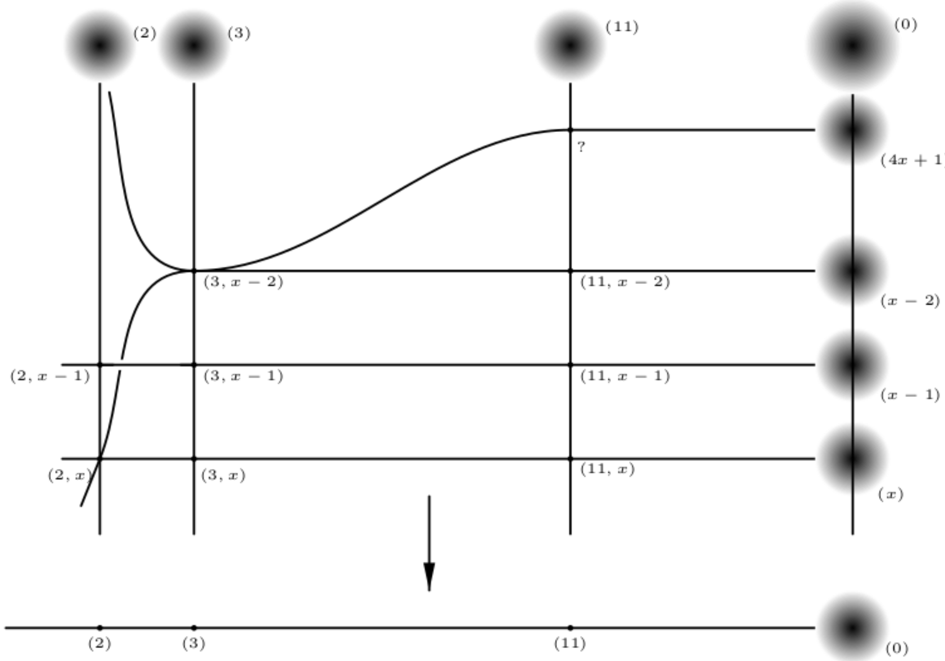

Figure 1: Spec $\mathbb{Z}[x]$ [EH, p. 85]

Example 6.1. Let **Hot** be the category of CW-complexes with $\mathrm{Hom}(X, Y)$ consisting of homotopy-classes of continuous maps from X to Y. Then if we set $z = \{*\}$, we get

$$\mathrm{Hom}_{\mathbf{Hot}}(z, X) = \pi_0(X),$$

so this is clearly not a faithful functor. ○

Example 6.2. Similarly in the category of schemes, we might try the final object $\mathrm{Spec}\,\mathbb{Z}$. But $\mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}\,\mathbb{Z}, X)$ has again no chance of being faithful. ○

6.2. THE FUNCTOR OF POINTS

The idea now is to consider not just a single object z, but the hom-sets from all objects of the category at once. To do this in a structurally sound way for a fixed object $X \in \mathbf{C}$, we consider the (contravariant) functor

$$h_X : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set},\ z \mapsto \mathrm{Hom}_{\mathbf{C}}(z, X).$$

That is we get an assignment

$$X \mapsto h_X \in \mathbf{Funct}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}).$$

[Some words about the category of functors. Full and faithful functors.] Thus we have a functor

$$h\colon \mathbf{C} \to \mathbf{Funct}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}), \quad X \mapsto h_X,$$

where morphisms in $\mathbf{C}$ are sent to the obvious natural transformations.

Proposition 6.3 (Yoneda Lemma)**.** *The functor h is fully faithful, i.e if X_1 and X_2 are any objects of $\mathbf{C}$, then h induces an isomorphism*

$$\mathrm{Hom}_{\mathbf{C}}(X_1, X_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Funct}}(h_{X_1}, h_{X_2}).$$

In particular, X_1 and X_2 are isomorphic if and only if h_{X_1} and h_{X_2} are.

Proof. Exercise. (Given $f: X_1 \to X_2$ evaluate the corresponding natural transformation $h_{X_1} \to h_{X_2}$ at X_1 and then at Id_{X_1}; and a similar construction the other way.) $\square$

Definition 6.4. A functor $F \in \mathbf{Funct}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set})$ is called *representable* if there is an object $X \in \mathbf{C}$ such that $F \cong h_X$.

Applied to schemes we obtain a fully faithful functor

$$\mathbf{Sch} \to \mathbf{Funct}(\mathbf{Sch}^{\mathrm{op}}, \mathbf{Set}).$$

We identify $\mathbf{Sch}$ with the corresponding full subcategory of $\mathbf{Funct}(\mathbf{Sch}^{\mathrm{op}}, \mathbf{Set})$. For a schemes X we then abuse notation and simply write X for the functor h_X. In particular, if Y is another scheme, we write

$$X(Y) = h_X(Y) = \mathrm{Hom}_{\mathbf{Sch}}(Y, X).$$

Moreover, for a ring R we set $X(R) = X(\mathrm{Spec}\, R)$. We call the elements of $X(Y)$ (resp. $X(R)$) *Y-valued points* (resp. *R-valued points*) of X. If k is a field, then we sometimes also call the elements of $X(k)$ *k-rational points* of X. Recall from the homework that $X(k)$ consists exactly of the points $x \in X^{\mathrm{top}}$ together with a field extension $k(x) \hookrightarrow k$.

Points in this sense are isomorphic with product: if X_1, X_2 and Y are any schemes, then

$$(X_1 \times X_2)(Y) = X_1(Y) \times X_2(Y).$$

This is just the universal property of the product.

Example 6.5. Let $X = \mathrm{Spec}\, \mathbb{Z}[x_1, \dots, x_n]/(f_1, \dots, f_m)$ for some polynomial $f_1, \dots, f_m \in \mathbb{Z}[x_1, \dots, x_n]$. Let R be a ring. Then an R-valued point is by definition a morphism

$$\mathrm{Spec}\, R \to \mathrm{Spec}\, \mathbb{Z}[x_1, \dots, x_n]/(f_1, \dots, f_m)$$

which is the same as ring homomorphism

$$\alpha\colon \mathbb{Z}[x_1,\dots,x_n]/(f_1,\dots,f_m) \to R.$$

But this is the same as choosing n elements $a_i = \alpha(x_i) \in R$ such that $f_j(a_1,\dots,a_n) = 0$ for $j = 1,\dots,m$. Thus R-valued points of X are the same as R-solutions of the system $f_1 = \cdots = f_m = 0$.

In particular $\mathbb{A}^n_{\mathbb{Z}}(R) = R^n$. ○

Since schemes are by definition covered by affine schemes, one can prove the following lemma.

Lemma 6.6. *Any scheme X is completely determined by the restriction of h_X to the subcategory of affine schemes. There is a fully faithful functor*

$$\begin{aligned}\mathbf{Sch} &\to \mathbf{Funct}(\mathbf{Ring}, \mathbf{Set})\\ X &\mapsto (R \mapsto \mathrm{Hom}_{\mathbf{Sch}}(\operatorname{Spec} R, X)).\end{aligned}$$

Thus one often identifies the category of schemes with the corresponding full subcategory of $\mathbf{Funct}(\mathbf{Ring}, \mathbf{Set})$. One calls a functor $F\colon \mathbf{Ring} \to \mathbf{Set}$ *representable* if it is in the essential image of the functor $\mathbf{Sch} \to \mathbf{Funct}(\mathbf{Ring}, \mathbf{Set})$.

We can of course do the same construction with the category $\mathbf{Sch}/S$ of schemes over a fixed scheme S. In this case $X(Y)$ consists of morphisms $Y \to X$ over S.

Example 6.7. Fix a ground field k_0 and consider schemes over k_0. Let k be a field extension of k_0. Then k-rational points of X are points $x \in X^{\mathrm{top}}$ together with an injection $k(x) \hookrightarrow k$ fixing the subfield k_0. In particular, if $k = k_0$ is is algebraically closed and X is of finite type, the $X(k)$ consists of exactly the closed points of X^{top} If X is integral, then these are the same as the points of the corresponding algebraic set.

More generally, let k be the algebraic closure of k_0 and X_0 a scheme over k_0. Set $X = X_0 \times_{\operatorname{Spec} k_0} \operatorname{Spec}(k)$. Then

$$\{k\text{-valued points of } X_0/k_0\} \cong \{k\text{-valued points of } X/k\} \cong \{\text{closed points of } X\}.$$

For this reason k-valued points of X_0 are often called *geometric points* of X. ○

Always be careful over which scheme you are working. For example, let $X = \operatorname{Spec} \mathbb{C}$. Then $X(\mathbb{C})$ is a single point in $\mathbf{Sch}/\mathbb{C}$. But $X(\mathbb{C}) = \mathrm{Aut}(\mathbb{C})$ in $\mathbf{Sch} = \mathbf{Sch}/\mathbb{Z}$.

Example 6.8. We can use the functor of points to give a scheme X additional structure, by requiring that h_X factors through some fixed functor from some other category of **Set**. For example, we say that G is a *group scheme* if we are given a factorization of h_G as

$$h_G\colon \mathbf{Ring} \to \mathbf{Grp} \to \mathbf{Set}.$$

In other words, a group scheme is a scheme G and a natural way of regarding $Hom(X, G)$ as a group for each X.

By the Yoneda lemma this induces a morphism $G \times G \to G$. However G^{top} is *not* a group! For example GL_n can be defined as

$$\operatorname{Spec} \mathbb{Z}[x_{ij}][\det(x_{ij})^{-1}],$$

the affine scheme of invertible integral $n \times n$ matrices. However one usually thinks of GL_n, as the functor that associates to each ring R the group $\mathrm{GL}_n(R)$. The point is that this family of groups already determines the structure of a scheme and the additional structure maps. ○

6.3. GEOMETRY OF FUNCTORS

One advantage of this point of view is that one can try to extend notions from geometry to arbitrary functors **Ring** → **Sch**, i.e. to functors which might not be representable by a scheme. We will see in a bit how this is useful in a bit, but first let us discuss some examples of this idea.

First we note that the category **Funct**(**Ring**, **Set**) (or indeed any category **Funct**(**C**, **Set**)) has fiber products: Let A, B and C be such functors and let $\alpha : A \to C$ and $\beta : B \to C$ be morphisms of functors (i.e. natural transformations). Then $A \times_C B$ is the functor which on on any object Z is defined to be

$$(A \times_C B)(Z) = A(Z) \times_{C(Z)} B(Z) = \{(a, b) \in A(Z) \times B(Z) : \alpha_Z(a) = \beta_Z(b) \text{ in } C(Z)\}.$$

To do any kind of geometry we need to define what an "open subset" is.

For this let $\alpha : G \to F$ be a natural transformation of functors **C** → **Set**. Then we α is called *injective* if for every object $X \in \mathbf{C}$ the induced map $\alpha_X : G(X) \to F(X)$ is injective. We will then say that G is a *subfunctor* of F. For example, if $U \subseteq X$ is a subscheme, then h_U is a subfunctor of h_X.

Definition 6.9. A subfunctor $\alpha : G \to F$ in **Funct**(**Ring**, **Set**) is an *open subfunctor* if, for each ring R and each map $\psi : h_{\operatorname{Spec} R} \to F$ the fiber product of functors

$$\begin{tikzcd} G_\psi \arrow[r] \arrow[d] & h_{SpecR} \arrow[d, "\psi"] \\ G \arrow[r, "\alpha"] & F \end{tikzcd}$$

the functor G_ψ is represented by a scheme and the map $G_\psi \to h_{\operatorname{Spec} R}$ corresponds via the Yoneda lemma to the inclusion of an open subscheme into $\operatorname{Spec} R$.

Similarly we can define *closed subfunctors*. Actually, we can get transport many other properties from morphisms of schemes to morphisms of maps.

Definition 6.10. A map $\alpha\colon G \to F$ of functors in **Funct**(**Ring**, **Set**) is called *representable* if for each ring R and each map $\psi\colon h_{\operatorname{Spec} R} \to F$ the fiber product functor $G_\psi = G \times_F h_{\operatorname{Spec} R}$

$$\begin{tikzcd} G_\psi \arrow[r] \arrow[d] & h_{SpecR} \arrow[d, "\psi"] \\ G \arrow[r, "\alpha"] & F \end{tikzcd}$$

the functor G_ψ is represented by a scheme.

Let $\mathscr{P}$ be a property of morphisms of schemes (eg. "proper", "separated", "smooth", ...). Then we say a map $\alpha\colon G \to F$ as above has property $\mathscr{P}$ if for each R and ψ as above the induced map $G_\psi \to \operatorname{Spec} R$ has property $\mathscr{P}$.

Two remarks are in order: Firstly, not every morphism of functors is representable, but there are morphisms $\alpha\colon G \to F$ where neither F nor G are representable, by α is. Secondly, this definition only really makes sense for properties $\mathscr{P}$ that are stable under base change and local on the base.

Similarly, we can define the notion of an *open covering* of a functor. This is a collection of open subfunctors that yields an open covering of a scheme whenever we pull back to a representable functor. More precisely, let $F\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ be a functor. Consider a collection $\{G_i \to F\}$ of open subfunctors of F. For each respresentable functor h_X and map $h_X \to F$, there are by definition open subschemes $U_i \subseteq X$ such that $h_X \times_F G_i \cong h_{U_i}$. Then $\{G_i \to F\}$ is an open covering if for any such map $h_X \to F$ the corresponding collection $\{U_i\}$ is an open covering of X.

6.4. PARAMETER AND MODULI SPACES

The main reason for doing this kind of "abstract nonsense" is that it is often much easier and more illuminating to define a space via its functor of points than it is to define the corresponding scheme. As a trivial example, we saw that it is very easy to define the fiber product of functors **Ring** → **Set**, while it is much more tedious to define the fiber product of schemes (where we actually skipped most of the definition). Similarly, it is easier to define GL_n as a functor rather than a sheaf.

Of course then one has to answer the following question: Given a functor $F\colon \mathbf{Ring} \to \mathbf{Set}$, is there a sheaf X such that $F \cong h_X$ (by the Yoneda Lemma, if such X exists, then it is unique up to isomorphism). This is called the *Grothendieck existence problem*. In general this is a very hard problem to solve.

There do however exist some criteria that help with showing that some functor is representable. They usually require to show that F satisfies some sort of sheaf condition and some sort of local condition.

What do we mean by sheaf condition? Here our viewpoint of sheaves as functors comes in handy. We say that a functor $F\colon \mathbf{Ring} \to \mathbf{Set}$ is a *sheaf in the Zariski topology* if for every ring R the restriction of F to open affine subsets of $\operatorname{Spec} R$ satisfies the usual sheaf condition. In other words, for every open covering of $X = \operatorname{Spec} R$ by distinguished open affines $U_i = \operatorname{Spec} R_{f_i}$ and every collection of elements $\varphi_i \in F(R_i)$ such that φ_i and φ_j map to the same element in $F(R_{f_if_j})$ (for all i, j), there is a unique element $\varphi \in F(R)$ mapping to each of the φ_i.

$$F(R) \to \prod_i (R_{f_i}) \rightrightarrows \prod_{i,j} F(R_{f_if_j}).$$

With that we have the following proposition (which essentially is just a restatement of the definition of a scheme, as being glued from affine schemes).

Proposition 6.11. *A functor $F\colon \mathbf{Ring} \to \mathbf{Set}$ is of the form h_X for some scheme X if and only if*

(i) F is a sheaf in the Zariski topology.

(ii) F is covered by affine schemes, i.e. there exist rings R_i and open subfunctors $\alpha_i\colon h_{\operatorname{Spec} R_i} \to F$ forming an open covering of F.

For example, to prove the existence of fiber products, on could check that:

(i) If $f\colon F \to H$ and $g\colon G \to H$ are two maps of functors, all of which are sheaves in the Zariski topology, then $F \times_H G$ is a sheaf in the Zariski topology.

(ii) Let $X \to S$ and $Y \to S$ be morphisms of schemes. Cover S by open affines $U_i = \operatorname{Spec} A_i$ and cover the inverse images of U_i in X and Y by open affines $\operatorname{Spec} A_{i\alpha}$ and $\operatorname{Spec} A_{i\beta}$ respectively. Then there is an open covering of the functor $h_X \times_{h_S} h_Y$ by the $\{h_{\operatorname{Spec} A_{i\alpha} \otimes_{A_i} A_{i\beta}}\}$.

Example 6.12. It is a non-trival fact that projective space $\mathbb{P}^n_{\mathbb{Z}}$ is represented by

$$h_{\mathbb{P}^n_{\mathbb{Z}}}(R) = \{\text{rank 1 direct summands of } R^{n+1}\}.$$

(Recall from commutative algebra, that a direct summand of a free module is projective. The rank of a projective R-module M at a prime $\mathfrak{p}$ is the rank of the free $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$. We require in the definition above that the rank is constant. In geometric terms, this means that we should think of M as a vector bundle on $\operatorname{Spec} R$.)

From this one can generalize and the Grassmannian as the functor

$$g = g(k, n)\colon \mathbf{Ring} \to \mathbf{Set}, \quad g(R) = \{\text{rank } k \text{ direct summands of } R^n\}.$$

To show that g is represented by a projective scheme $\operatorname{Grass}_{\mathbb{Z}}(k, n)$ one can proceed as follows. First show that g is a sheaf in the Zariski topology. The affine cover is then obtained similarly to the case of algebraic sets: Define a natural transformation $g \to h_{\mathbb{P}^r_{\mathbb{Z}}}$ with $r = \binom{n}{k} - 1$ by sending a summand $M \subseteq R^n$ to $\bigwedge kM \subseteq \bigwedge kR^n$. Cover $\mathbb{P}^r$ by the standard open affines U_i (whose functor of points can be described). Explicitly compute $g \cap h_{U_i} = g \times_{h_{\mathbb{P}^r}} h_{U_i}$ and show that these functors are represented by affine schemes. ○

Example 6.13. One often wants to "parametrize" certain geometric objects by a scheme. An example of that idea that you will see quite often is the following. Fix a polynomial P and a natural number n. We would like to parametrize subschemes of $\mathbb{P}^n$ with Hilbert polynomial P. That is we consider the functor h_P whose value at a field k is

$$h_P(k) = \{\text{closed subschemes of } \mathbb{P}^n_k \text{ with Hilbert polynomial } P\}.$$

(This of course assumes a suitable generalization of Hilbert polynomials.) We can extend this to a functor on all schemes by

$$h_P(S) = \{\text{closed subschemes } X \subseteq \mathbb{P}^n_S, \text{ flat over } S, \text{ whose fibers over points of } S \text{ have Hilbert polynomial } P\}.$$

Here $\mathbb{P}^n_S = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} S$ and "flat" is a technical condition that essentially says that the fibers of X over S vary nicely.

It turns out that h_P can be represented by a scheme Hilb^P. Not only do we get such a (very useful) scheme, we also get an additional bonus. Let $X \subseteq \mathbb{P}^n_{\mathrm{Hilb}^P}$ be the subscheme corresponding to the identity map. Let $S \to \mathrm{Hilb}^P$ be a morphism (i.e. an element of $h_P(S)$, corresponding to closed subscheme $Y \subseteq \mathbb{P}^n_S$. Then we can recover Y as a fiber product $Y = X \times_{\mathrm{Hilb}^P} S \subseteq \mathbb{P}^n_{\mathbb{Z}} \times S$. We call X a *universal family*. ○

Example 6.14. We could get even more ambitious and ask whether there is a parameter space for all non-singular curves of a fixed genus g over a field k. That is we consider the functor M_g that assigns to any k-scheme S the set of all flat morphisms $\pi : X \to S$ whose fibers are non-singular curves of genus g, up to isomorphism $X \cong X'$ of schemes over B.

Unfortunately it turns out that there is no scheme that represents the functor M_g. The problem here (and in related questions) is that there exists curves with non-trivial automorphisms. There are essentially three ways of dealing with this problem:

(i) Only consider curves without non-trivial automorphisms.

(ii) Consider the scheme whose functor of points "most closely approximates" M_g. This is called the *coarse moduli space*.

(iii) Invent a theory that can deal with non-trivial automorphisms. This has been done and leads to *algebraic stacks*.

○

For more information about the idea of moduli spaces and their importance, I recommend the article [B], which is also available from `https://www.ma.utexas.edu/users/benzvi/math/pcm0178.pdf`.

7. SOME NOTES ON THE LITERATURE

The next topic in a standard course of algebraic geometry is the theory of $\mathcal{O}_X$-modules and coherent sheaves. You should be able to pick up any text book on algebraic geometry in start with the chapter on coherent sheaves (or $\mathscr{O}_X$-modules or sheaves of modules), maybe with occasional backtracking to fill in some details.

The standard text book on algebraic geometry is of course [H]. It develop the theory of schemes is quite some detail and is generally very well written, though it does lack motivational exposition. It should be noted however, that many important definitions and results are only given as exercises.

Another highly regarded book is [M]. Note that in this book what is now usually called a "scheme" is called a "pre-scheme", while "scheme" refers to what is a "separated scheme" in our terminology.

The authors of both of these books are mainly interested in "classical" algebraic geometry, i.e. geometry over an algebraically closed field. A more recent book that treats the theory with more of a perspective on arithmetic geometry in [L].

The book [EH] focuses more on examples and is a good complement to any of the more technical treatments above.

Finally, the ultimate reference for anything about schemes is the series *Eléments de géométrie algébrique I-IV* by Grothendieck. Structured in more of a "Bourbaki" style it is not a readable textbook, but it probably contains all you might ever want to know about the general theory of schemes (and quite a bit more).

In a different direction, the book [V] is a very readable introduction to algebraic geometry over the complex numbers, and in particular to Hodge theory.

Lastly, I can recommend reading [S]. While it doesn't discuss the general theory very much, it shows how the different aspects of geometry interact. Understanding the example of elliptic curves is something that every algebraic needs to do sooner or later.

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