MATH 532 – HOMEWORK SET 4

MARCH 14, 2016

In the first two problems we consider quasi-projective algebraic sets. In the remaining problems we consider schemes.

1) Let $X = \{x_1^3 = x_2^2\} \subseteq \mathbb{A}^2$ and consider the blowup $\pi : \tilde{X} \to X$ of X at the origin. Compute the exceptional locus of π and show that $\tilde{X}$ is non-singular.

2) Let $X = \{x_1^2 + x_2^2 = x_3^2\} \subseteq \mathbb{A}^3$ be the cone. Show that X is isomorphic to $Y = \{x_1^2 - x_2x_3 = 0\}$ and show that Y has exactly one singularity (at the origin). Let $\tilde{Y} \to Y$ be the blowup at the origin. Show that $\tilde{Y}$ is non-singular. Further, show that the exceptional locus of $\tilde{Y}$ is isomorphic to $\mathbb{P}^1$. [Hint: one way to show this is to use the map $(a : b) \mapsto (ab : a^2 : b^2)$.]

3) Let X be a scheme and let K be any field. Show that to give a morphism $\operatorname{Spec} K \to X$ it is equivalent to give a point $x \in X$ and an inclusion map (of fields) $k(x) \to K$.

4) Let X be a scheme. For any point $x \in X$ define the *Zariski tangent space* T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k and let $k[\varepsilon]/(\varepsilon^2)$ be the *ring of dual numbers* over k. Show that to give a k-morphism[1] $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \to X$ is equivalent to giving a point $x \in X$, such that $k(x) = k$, and an element of T_x.

[1] That is, a morphism of schemes over $\operatorname{Spec} k$.