MATH 532 – HOMEWORK SET 2

FEBRUARY 1, 2016

1) (a) Let X be a topological space. Show that continuous functions form a sheaf on X.

   (b) Show that bounded functions form a presheaf, but not a sheaf on $\mathbb{C}$ (with the usual topology).

2) A morphism of (pre-)sheaves is a natural transformation of the corresponding functors. Thus if $\mathscr{F}$ and $\mathscr{G}$ are sheaves (with values in a category **C**) on a topological space X, then a morphism $\varphi\colon \mathscr{F} \to \mathscr{G}$ consists of a collection of maps (in **C**) $\varphi_U\colon \mathscr{F}(U) \to \mathscr{G}(U)$ such that for all inclusions $V \subseteq U \subseteq X$ of open sets the diagram

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi_U} & \mathscr{G}(U) \\ \downarrow{\scriptstyle \rho^U_V} & & \downarrow{\scriptstyle \rho^U_V} \\ \mathscr{F}(V) & \xrightarrow{\varphi_V} & \mathscr{G}(V) \end{array}$$

commutes.

   (a) Show that for each $x \in X$ the morphism φ induces a map on stalks $\varphi_x\colon \mathscr{F}_x \to \mathscr{G}_x$.

   (b) Let **C** be an abelian category (you can consider sheaves of abelian groups for concreteness). Define the kernel of φ by $U \mapsto \ker \varphi_U$. Show that this defines a sheaf on X.

   (c) Show that $\ker \varphi = 0$ if and only if $\ker \varphi_x = 0$ for all $x \in X$.

3) (a) Let $X = \{a, b\}$ be given a topology such that the open sets are $\emptyset, \{a\}, X$. Explain why giving a sheaf of vector spaces on X is equivalent to giving two vector spaces V, W and a linear map between them. Describe morphisms of sheaves in a similar way.

   (b) Let $X = \{a, b\}$ have the discrete topology where all subsets are open. Describe the minimal data needed to define a sheaf of vector spaces on X. What are morphisms of sheaves?

4) Let $f : X \to Y$ be any map between algebraic sets. Assume that there exists an open cover $\{U_i : i \in I\}$ of X such that the restrictions $f|_{U_i} : U_i \to Y$ are morphisms. Show that f is a morphism.

5) Let $f, g\colon X \to Y$ be two morphisms of quasi-projective algebraic sets. Suppose that there exists a dense open subset U such that $f|_U = g|_U$. Show that $f = g$ on X.