MATH 532 – HOMEWORK SET 1

JANUARY 15, 2016

1) (a) Recall that a topological space X is called (quasi-)compact if every open cover of X has a finite subcover. Show that every Noetherian topological space is quasi-compact.

   (b) Let X be a topological space and assume that $X = \bigcup_{i=1}^n U_i$ is a finite open cover such that each U_i is a Noetherian topological space (in the induced topology). Show that X is Noetherian. Conclude that $\mathbb{P}^n$ is a Noetherian topological space.

2) Show that every morphism of affine algebraic sets is continuous with respect to the Zariski topology. Give an example of a continuous map of affine algebraic sets that is not a morphism.

3) Show that any homogeneous ideal of $k[x_1, \dots, x_n]$ can be generated by finitely many homogeneous polynomials.

4) Let $(U_i)_{i=0,\dots,n}$ be the standard open cover of $\mathbb{P}^n$ by affine subsets and write $\varphi_i \colon U_i \xrightarrow{\sim} \mathbb{A}^n$ for the identification discussed in class.

   (a) Describe the change-of-coordinates map $\varphi_j \circ \varphi_i^{-1} \colon \mathbb{A}^n \to \mathbb{A}^n$.

   (b) Consider the affine algebraic set $X = \{x_2^2 = x_1^3 - x_1\} \subseteq \mathbb{A}^2$. Describe the projective closure $\overline{X}$ of X, i.e. the projective algebraic set $\overline{\varphi_0^{-1}(X)} \subseteq \mathbb{P}^2$. Further give the equations of $\varphi_1(\overline{X} \cap U_1)$ and $\varphi_2(\overline{X} \cap U_2)$.

5) Consider the morphism $f \colon \mathbb{A}^1 \to \mathbb{A}^3$ given by $t \mapsto (t, t^2, t^3)$. Show that the image $X = f(\mathbb{A}^1)$ is a closed subset of $\mathbb{A}^3$ and describe its ideal. Describe the closure of X in $\mathbb{P}^3$ and its homogeneous ideal.

6) Show that $\mathbb{P}^1 \times \mathbb{P}^1$ can be viewed as a projective algebraic subset of $\mathbb{P}^3$, i.e. give a bijection of $\mathbb{P}^1 \times \mathbb{P}^1$ with a closed subset of $\mathbb{P}^3$.