

MATH 532 – FINAL PROBLEMS

DUE: APRIL 25

Do six of the following problems. If you hand in solutions to more than six problems, I will count the best six. The solutions are due by the end of April 25. If you have typed solutions, you can send them to me by e-mail. Otherwise please leave them in my mailslot at the department or under my office door.

ALGEBRAIC SETS

In the following three problems all spaces are assumed to be quasi-projective algebraic sets over an algebraically closed ground field.

Problem 1. Let $f: X \rightarrow Y$ be a dominant morphism of irreducible quasi-projective algebraic sets. Show that there is a non-empty open subset U of Y such that every component of the fiber $f^{-1}(P)$ has dimension $\dim X - \dim Y$ for all $P \in U$. (Hint: Reduce to $X \subseteq \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ affine. By considering the graph $(P, f(P)) \in \mathbb{A}^{n+m}$, reduce to the case where $f: \mathbb{A}^{n+1} \rightarrow \mathbb{A}^n$ is the projection map.)

From this one can deduce that for a surjective proper map $f: X \rightarrow Y$ one has

$$\dim f^{-1}(y) \geq \dim X - \dim Y, \quad y \in Y,$$

with equality on a non-empty open subset of Y . You may use this result in your solution to the next problem.

Problem 2. Let X be an irreducible quasi-projective algebraic set and $Y \subseteq X \times \mathbb{P}^n$ a closed subset, $f: Y \rightarrow X$ the projection to the first factor. Assume that all fibres of f are irreducible and have the same dimension. Prove that Y is irreducible. (Hint: prove that every component of Y is a union of fibres of f . Then study the images of the components.)

Problem 3. For given $n, d > 0$ let M_0, M_1, \dots, M_N (with $N = \binom{n+d}{n} - 1$) be all the monomials of degree d in the $n + 1$ variables x_0, \dots, x_n . Define a map $\rho_d: \mathbb{P}^n \rightarrow \mathbb{P}^N$ by sending a point $a = (a_0 : \dots : a_n)$ to $(M_0(a) : \dots : M_N(a))$.

- (i) Show that ρ_d is a homeomorphism of \mathbb{P}^n with an irreducible closed subset of \mathbb{P}^N . (Hint: Consider the kernel of the homomorphism $\theta: k[y_0, \dots, y_N] \rightarrow k[x_0, \dots, x_n]$ defined by $y_i \mapsto M_i$.)
- (ii) Compute the degree of the image of ρ_d .

RESOLUTION OF SURFACE SINGULARITIES

In the following two problems all spaces are quasi-projective algebraic sets over an algebraically closed field of characteristic zero. You are allowed to use the result of Exercise 9.22 in Gathmann's notes.

We can sometimes resolve the singularities of a surface by blowing up singular points and taking the strict transform. Suppose X has only one singular point P . Then in the resolution $f: X' \rightarrow X$ we can consider the inverse image $f^{-1}(P)$, consisting of a finite union of irreducible curves. The *resolution graph* of f is constructed as follows. Take one node for each component of $f^{-1}(P)$ and connect two nodes with an edge if the two components intersect.

Problem 4. Let X have the A_k singularity:

$$X = Z(x^{k+1} + y^2 + z^2) \subseteq \mathbb{A}^3,$$

where $k \geq 1$. Find the resolution of X by a sequence of blowups of points. Find the graph of the resolution. (Hint: After one blowup the inverse image of the singular point consists of two irreducible curves. In the next blowup the strict transforms of these curves do not lie in the chart containing the singularity. However, in another chart, the strict transforms should be disjoint, each meeting one of the two new exceptional curves.)

Problem 5. Let X have the D_k singularity:

$$X = Z(x^{k-1} + xy^2 + z^2) \subseteq \mathbb{A}^3,$$

where $k \geq 4$. Find the resolution of X when $k = 4$ by a sequence of blowups of points. Find the graph of the resolution. (Hint: After the first blowup the inverse image of 0 consists of one irreducible curve. The remaining singular points lie on this curve. When we blow up a point that is not the origin using our usual method, we first have to change coordinates so that it becomes the origin.)

SCHEMES

In the remaining problems, all spaces are schemes.

Problem 6. Show that $\text{Spec } \mathbb{Z}[x]$ consists of the following points:

- (i) (0) ;
- (ii) (p) , for $p \in \mathbb{Z}$ prime;
- (iii) principal ideals of the form (f) , where $f \in \mathbb{Z}[x]$ is a polynomial irreducible over \mathbb{Q} whose coefficients have greatest common divisor 1; and
- (iv) maximal ideals of the form (p, f) , where $p \in \mathbb{Z}$ is a prime and $f \in \mathbb{Z}[x]$ a monic polynomial whose reduction mod p is irreducible.

Describe the closure of each non-closed point. (Hint: Consider the map $\text{Spec } \mathbb{Z}[x] \rightarrow \text{Spec } \mathbb{Z}$.)

Problem 7. Let (X, \mathcal{O}_X) be a ringed space and G a group acting on X . Then the quotient X/G can again be given the structure of a ringed space. Let X/G be the quotient space, that means, the set of G -orbits. This set is given the quotient topology where $U \subseteq X/G$ is open if and only if $\pi^{-1}(U)$ is open in X . Here $\pi: X \rightarrow X/G$ is the quotient map. The sheaf of rings on X/G consists of G -invariant sections of \mathcal{O}_X :

$$\mathcal{O}_{X/G}(U) = \mathcal{O}_X(\pi^{-1}(U))^G.$$

Note that if X is covered by G -invariant open sets V_i , then X/G is covered by the quotients V_i/G .

Let k be a field.

- (i) Let $X = \mathbb{A}_k^2 - \{(0, 0)\}$, let $G = k^*$, and let G act by

$$t \cdot (x, y) = (tx, ty), \quad t \in k^*, (x, y) \in X.$$

Show that $X/G = \mathbb{P}_k^1$. (Hint: show that X is covered by two G -invariant affines, isomorphic to $\mathbb{A}_k^1 \times G$. Describe the quotients of these charts and how the quotients are glued in X/G .)

- (ii) Let X and G be as in the previous part, but let the action be

$$t \cdot (x, y) = (tx, t^{-1}y), \quad t \in k^*, (x, y) \in X.$$

Show that X/G is the line \mathbb{A}_k^1 with doubled origin.

Problem 8. In this problem we consider schemes over a fixed finite field \mathbb{F}_q . The zeta function of a scheme X is defined as

$$\zeta(X, s) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} q^{-ms}\right),$$

where $N_m = \#X(\mathbb{F}_{q^m})$ is the number of \mathbb{F}_{q^m} -rational points of X , i.e. the number of morphisms $\text{Spec } \mathbb{F}_{q^m} \rightarrow X$ over \mathbb{F}_q .

- (i) By explicit computations, show that $\zeta(\mathbb{A}_{\mathbb{F}_q}^1, s)$ and $\zeta(\mathbb{P}_{\mathbb{F}_q}^1, s)$ are rational functions in q^{-s} . (Hint: You will need the power series for $\log(1-x)$.)
- (ii) It is known that for an elliptic curve E (i.e., a smooth cubic curve in $\mathbb{P}_{\mathbb{F}_q}^2$)

$$\zeta(E, s) = \frac{(1 - \alpha q^{-s})(1 - \beta q^{-s})}{(1 - q^{-s})(1 - q^{1-s})},$$

where $\alpha, \beta \in \mathbb{C}$ are complex conjugate numbers with absolute value \sqrt{q} . Show that

$$|N_1 - (q + 1)| \leq 2\sqrt{q}.$$

Note: You can ignore any convergence questions.