Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

Consider the function, $h(x) = 2x^3 - 9x^2 + 12x$.

(a) What are the coordinates of the local maximum of $h(x)$?

Answer: $(1, 5)$

Solution: The function $h(x)$ has critical points, but no singular points since its derivative $h'(x) = 6x^2 - 18x + 12$ is defined for all values of x. The critical points of $h(x)$ are computed by equating $h(x) = 0$ which yields

$$6x^2 - 18x + 12 = 0, \text{ i.e. } 6(x^2 - 3x + 2) = 6(x-1)(x-2) = 0,$$

and thus $x = 1$ and $x = 2$. Using either the Second Derivative Test, i.e. computing $h''(x) = 12x - 18$ and then plugging in the critical numbers $x = 1$ and $x = 2$ in $h''(x)$, or by simply noticing that $h'(x)$ changes from positive to negative at $x = 1$ and from negative to positive at $x = 2$, we conclude that $x = 1$ is a point of local maximum, while $x = 2$ is a point of local minimum. We compute $f(1) = 5$ and $f(2) = 4$.

(b) What are the coordinates of the local minimum of $h(x)$?

Answer: $(2, 4)$

Short answer questions — you must show your work

2. 8 marks Each part is worth 2 marks.

(a) Find the intervals where $f(x) = \frac{x^2}{x-3}$ is decreasing.

Solution: First of all, $f(x)$ is defined for all $x \neq 3$. In order to find where is $f(x)$ decreasing, we find where is $f'(x)$ negative. So,

$$f'(x) = \frac{2x(x-3) - x^2 \cdot 1}{(x-3)^2} = \frac{2x^2 - 6x - x^2}{(x-3)^2} = \frac{x(x-6)}{(x-3)^2}.$$

and thus, since the denominator is always positive, we conclude that $f'(x) < 0$ when $x(x-6) < 0$. Recalling that the domain of definition for $f(x)$ is $x \neq 3$, we conclude that $f(x)$ is decreasing on the intervals $(0, 3)$ and $(3, 6)$.

Marking scheme:

- 1 mark for computing $f'(x)$ in *simplified form*, i.e. $f'(x) = \frac{x(x-6)}{(x-3)^2}$.
- 1 mark for writing the correct intervals where $f(x)$ is increasing; it is acceptable also $[0, 3) \cup (3, 6]$, but **not** acceptable $(0, 6)$.

(b) Let $f(x) = \cos(x^3 + x^2 - 2)\sin(2x)$. Show that there exists a real number c such that $f'(c) = 0$.

Solution: We note that $f(0) = f(\pi) = 0$. Then using the Mean Value Theorem (note that the function is differentiable for all real numbers), we get that there exists $c \in (0, \pi)$ such that

$$f'(c) = \frac{f(2\pi) - f(0)}{\pi - 0} = 0.$$

(Alternatively, Rolle's Theorem gives the same conclusion.) **Marking scheme:**

- 1 mark for writing that $f(0) = f(\pi) = 0$.
- 1 mark for invoking Mean Value Theorem or Rolle's Theorem correctly to conclude the existence of $c \in (0, 2\pi)$ such that $f'(c) = 0$.

(c) Evaluate $\lim_{x\to\infty} \frac{\arctan x - \frac{\pi}{2}}{1/x}$.

Solution: This limit is a $\frac{0}{0}$-indeterminate form with differentiable numerator and denominator. So we can use de L'Hospital's rule:

$$\begin{aligned}\lim_{x\to\infty} \frac{\arctan x - \frac{\pi}{2}}{1/x} &= \lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{-x^{-2}} \\ &= \lim_{x\to\infty} \frac{-x^2}{1+x^2} \\ &= \lim_{x\to\infty} \frac{-1}{\frac{1}{x^2}+1} = -1.\end{aligned}$$

(d) Evaluate $\lim_{x\to 0^+} \log(x)\tan(x)$.

Solution: This limit is a "$\infty \cdot 0$"-indeterminate form with differentiable numerator and denominator. So we rewrite it as a $\frac{\infty}{\infty}$-indeterminate form:

$$\lim_{x\to 0^+} \log(x)\tan(x) = \lim_{x\to 0^+} \frac{\log(x)}{(\tan(x))^{-1}}.$$

Now we can use de L'Hospital's rule:

$$\begin{aligned}\lim_{x\to 0^+} \frac{\log(x)}{(\tan(x))^{-1}} &= \lim_{x\to 0^+} \frac{x^{-1}}{-(\tan(x))^{-2}\sec^2(x)} \\ &= \lim_{x\to 0^+} -\frac{1}{x\frac{\cos^2(x)}{\sin^2(x)}\frac{1}{\cos^2}(x)} \\ &= \lim_{x\to 0^+} -\frac{1}{x\frac{1}{\sin^2(x)}} \\ &= \lim_{x\to 0^+} -\frac{\sin^2(x)}{x} \\ &= \lim_{x\to 0^+} -\sin(x)\frac{\sin(x)}{x} = -0 \cdot 1 = 0.\end{aligned}$$