Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

   Consider a function, $h(x)$, whose third Maclaurin polynomial is $-x + 2x^2 + \frac{2}{3}x^3$.

   (a) What is $h^{(3)}(0)$?

   Answer: 4

   Solution: The third Maclaurin polynomial for $h(x)$ is

   $$h(0) + h'(0)x + \frac{h''(0)}{2} \cdot x^2 + \frac{h^{(3)}(0)}{6} \cdot x^3 = -x + 2x^2 + \frac{2}{3}x^3.$$

   Thus $h^{(3)}(0) = 6 \cdot \frac{2}{3} = 4$ and for the next part, we note that $h(0) = 0$.

   (b) What is $h(0)$?

   Answer: 0

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

   (a) Find the global maximum and the global minimum for $f(x) = 2x^3+9x^2+2$ on the interval $[-4, -1]$.

   Solution: We compute $f'(x) = 6x^2 + 18x$, which means that $f(x)$ has no singular points (i.e., it is differentiable for all values of x), but it has two critical points obtained by solving $f'(x) = 0$, i.e. $6x(x-3) = 0$ which yields the two critical points $x = 0$ and $x = 3$. In order to compute the global maximum and the global minimum for $f(x)$ on the interval $[-4, -1]$, we compute

   $$f(-4) = 18,\ f(-3) = 29 \text{ and } f(-1) = 9.$$

   So, the global maximum is $f(-3) = 29$ while the global minimum is $f(-1) = 9$.

   (b) Consider a function $f(x)$ which has $f'''(x) = \dfrac{e^x}{4-x}$. Show that when we approximate $f(0)$ using its second Taylor polynomial around $x = -1$, the absolute error is less than $\frac{1}{20} = 0.05$.

   Solution:

   - The error is bounded (in absolute value) by

   $$\max_{c\in[-1,0]} \left| \frac{f'''(c)}{3!} \cdot (0-(-1))^3 \right| = \max_{c\in[-1,0]} \left| \frac{e^c}{6(4-c)} \right|.$$

   - Since $c \in [-1, 0]$, we know that $\left|\frac{e^c}{6(4-c)}\right| = \frac{e^c}{6(4-c)}$ since both numerator and denominator are positive.

- When $-1 \le c \le 0$, we know that $e^{-1} \le e^c \le e^0 = 1$ and $5 \le 4 - c \le 4$, and that numerator and denominator are non-negative, so

$$\left|\frac{e^c}{6(4-c)}\right| = \frac{e^c}{6(4-c)} \le \frac{1}{6(4-c)} \le \frac{1}{6 \cdot 4}$$
$$= \frac{1}{24} \le \frac{1}{20}$$

as required.

- Alternatively, notice that e^c is an increasing function of c, while $4 - c$ is a decreasing function of c. Hence the fraction is an increasing function of c and takes its largest value at $c = 0$. Hence

$$\left|\frac{e^c}{6(4-c)}\right| \le \frac{1}{6 \times 4} = \frac{1}{24} \le \frac{1}{20}.$$

Marking scheme:

- 1 mark for writing that the error is bounded (in absolute value) by

$$\max_{c \in [-1,0]} \left|\frac{f'''(c)}{3!} \cdot (0 - (-1))^3\right| = \max_{c \in [-1,0]} \left|\frac{e^c}{6(4-c)}\right|.$$

- 1 mark for explaining why $c = 0$ is the right choice and then verifying that in that case the error is still bounded above by 0.05.
- **The students lose 1 mark if they don't explain why $c = 0$ is the right choice (and they simply plug in $c = 0$, or compare the values they get between plugging in $c = 0$ and $c = -1$).**

Long answer question — you must show your work

3. [4 marks] A 20m long extension ladder leaning against a wall starts collapsing at a rate of 2m/s, while the foot of the ladder remains a constant 5m from the wall. How fast is the ladder moving down the wall after 3.5 seconds?

Solution:

- If we write $z(t)$ for the length of the ladder at time t and $y(t)$ for the height of the top end of the ladder at time t we have

$$z(t)^2 = 5^2 + y(t)^2.$$

- We differentiate the above equation with respect to t and get

$$2z \cdot z' = 2y \cdot y',$$

- We are told that $z'(2.5) = -2$ and $z(2.5) = 20 - 3.5 \cdot 2 = 13$.
- At this point $y = \sqrt{z^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12$.
- Hence

$$\begin{aligned} 2 \cdot 13 \cdot (-2) &= 2 \cdot 12 y' \\ y' &= -\frac{2 \cdot 13}{12} = -\frac{13}{6} \text{ meters per second.} \end{aligned}$$

Marking scheme:

- 1 mark for obtaining the equation $2z(t) \cdot z'(t) = 2x \cdot x'$.
- 1 mark for $z' = -2$, $z = 13$ all correct.
- 1 mark for computing $y(2.5) = 12$.
- 1 mark for obtaining the correct answer $y'(2.5) = -\frac{13}{6}$m/s.