First Name: ______________ Last Name: ______________

Student-No: ______________ Section: ______________

Short answer questions — you must show your work

1. 6 marks Each part is worth 2 marks.

(a) For what values of x does the derivative of $\dfrac{e^x + x}{\cos(x)}$ exist? Explain your answer.

Solution: The function is differentiable whenever $\cos(x) \neq 0$ since the derivative equals

$$\frac{(e^x + 1) \cdot \cos(x) - (e^x + x) \cdot (-\sin(x))}{(\cos(x))^2},$$

which is well-defined unless $\cos(x) = 0$. So, the function is differentiable for all real values x except for $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$.

Marking scheme:

- 1 mark for writing that the function is differentiable whenever the denominator is nonzero.
- 1 mark for solving correctly and finding **all** points where the function is not differentiable.

(b) Consider a function of the form $f(x) = Ae^{kx}$ where A and k are constants. If $f(0) = 4$ and $f(3) = 1$, find the constants A and k.

Solution: The two pieces of information give us

$$f(0) = A = 4 \qquad\qquad f(3) = Ae^{3k} = 1$$

Thus we know that $A = 4$ and so $f(3) = 1 = 4e^{3k}$. Hence

$$\begin{aligned} e^{3k} &= \frac{1}{4} \\ 3k &= \log(1/4) \\ k &= \frac{1}{3} \cdot \log(1/4) = -\frac{\log(4)}{3}. \end{aligned}$$

Marking scheme:

- One mark for each correct constant. The students do not need to simplify $\log(1/4) = -\log(4)$.

(c) Estimate $\sin\left(\frac{13\pi}{12}\right)$ using a linear approximation

Solution: We use the function $f(x) = \sin(x)$ and point $a = \pi$ as the centre of our approximation since we know that

$$\sin(a) = f(\pi) = \sin \pi = 0.$$

We compute $f'(x) = \cos(x)$; so

$$f'(\pi) = \cos(\pi) = -1.$$

So, the linear approximation of $\sin\left(\frac{13\pi}{12}\right) = f\left(\frac{13\pi}{12}\right)$ is

$$T_1\left(\frac{13\pi}{12}\right) = f(\pi) + f'(\pi) \cdot \left(\frac{13\pi}{12} - \pi\right) = 0 + (-1) \cdot \frac{\pi}{12} = -\frac{\pi}{12}.$$

Marking scheme:

- 1 mark for writing that the function to use for the linear approximation is $f(x) = \sin(x)$, that $a = \pi$ and that the formula for the linear approximation is $f(a) + f'(a) \cdot (x - a)$.
- 1 mark for obtaining the linear approximation $-\frac{\pi}{12}$.

Long answer question — you must show your work

2. [4 marks] Determine whether the derivative of following function exists at $x = 0$

$$f(x) = \begin{cases} x^3 \sin\left(\dfrac{1}{x}\right) & \text{if } x < 0 \\ 6x^3 + 2xe^x & \text{if } x \geq 0 \end{cases}$$

You must justify your answer.

Solution: The function is differentiable at $x = 0$ if the following limit:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x) - 0}{x} = \lim_{x\to 0} \frac{f(x)}{x}$$

exists (note that we used the fact that $f(0) = 0$ as per the definition of the second branch which includes the point $x = 0$). We compute left and right limits; so

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{6x^3 + 2xe^x}{x} = \lim_{x\to 0^+} 6x^2 + 2e^x = 2$$

and

$$\lim_{x\to 0^-} \frac{x^3 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0^-} x^2 \cdot \sin\left(\frac{1}{x}\right).$$

This last limit equals 0 by Squeeze Theorem since

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

and so,

$$-x^2 \leq x^2 \cdot \sin\left(\frac{1}{x}\right) \leq x^2,$$

where in these inequalities we used the fact that $x^2 \geq 0$. Finally, since $\lim_{x\to 0^-} -x^2 = \lim_{x\to 0^-} x^2 = 0$, Squeeze Theorem yields that also $\lim_{x\to 0^-} x^2 \sin\left(\frac{1}{x}\right) = 0$, as claimed.

Since the left and right limits do not match, we conclude that indeed $f(x)$ is *not* differentiable at $x = 0$. **Marking scheme:**

- 1 mark for writing the condition for differentiability, i.e. that the limit $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x}$ must exist.
- 1 mark for computing the right limit correctly.
- 1 mark for computing the left limit correctly. They lose 1 mark if they don't explain the use of Squeeze Theorem.
- 1 mark for the correct answer (i.e. both left and right limits do not match for $\lim_{x\to 0} f(x)/x$, hence the function is not differentiable at $x = 0$).