First Name: ____________ Last Name: ____________

Student-No: ____________ Section: ____________

Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your **simplified** answers in the boxes.
Marking scheme: 1 for each correct, 0 otherwise

(a) Differentiate $\log\left(\sqrt{2+\cos(x)}\right)$. Recall that $\log x = \log_e x = \ln x$.

Answer: $\dfrac{-\sin(x)}{2(2+\cos(x))}$

Solution: This requires us to apply the chain rule twice.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\log\left(\sqrt{2+\cos(x)}\right)\right] &= \frac{1}{\sqrt{2+\cos(x)}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{2+\cos(x)}\right] \\ &= \frac{1}{\sqrt{2+\cos(x)}}\cdot\frac{1}{2\sqrt{2+\cos(x)}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left[2+\cos(x)\right] \\ &= \frac{1}{\sqrt{2+\cos(x)}}\cdot\frac{1}{2\sqrt{2+\cos(x)}}\cdot(-\sin(x)) \\ &= \frac{-\sin(x)}{2(2+\cos(x))}.\end{aligned}$$

(b) If $x^2+y^3=xy+3$ compute $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y-2x}{3y^2-x}$

Solution: Differentiate the equation

$$\begin{aligned}2x+3y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= y + x\frac{\mathrm{d}y}{\mathrm{d}x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y-2x}{3y^2-x}\end{aligned}$$

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

(a) Differentiate $f(x) = (3\sin(x))^{1/x}$.

Solution: We use logarithmic differentiation; so

$$\log f(x) = \frac{1}{x}\log(3\sin(x))$$

Then differentiate to obtain

$$\frac{f'(x)}{f(x)} = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\log(3\sin(x))}{x} = \frac{\frac{3\cos(x)}{3\sin(x)} \cdot x - \log(3\sin(x))}{x^2}.$$

In conclusion:

$$f'(x) = f(x) \cdot \frac{\frac{x\cos(x)}{\sin(x)} - \log(3\sin(x))}{x^2} \quad \text{or equivalently}$$
$$= (3\sin(x))^{1/x} \frac{\frac{x\cos(x)}{\sin(x)} - \log(3\sin(x))}{x^2}$$

Marking scheme:

- 1 mark for using logarithmic differentiation, i.e. writing that $\log(f(x)) = \frac{1}{x}\log(3\sin(x))$ — provided they also write that the derivative of $\log f(x)$ is $f'(x)/f(x)$. It is not enough just to write "Use log-diff (or similar)" they have to get $f'(x)/f(x) = \mathit{blah}$ or $f'(x) = f(x) \times \mathit{blah}$.
- 1 mark for correct derivative for $f(x)$; accept as correct answer also $f'(x) = f(x) \cdot \mathit{blah}$.

(b) For what values of x does the derivative of $\dfrac{e^x + x}{\cos(x)}$ exist? Explain your answer.

Solution: The function is differentiable whenever $\cos(x) \neq 0$ since the derivative equals

$$\frac{(e^x + 1) \cdot \cos(x) - (e^x + x) \cdot (-\sin(x))}{(\cos(x))^2},$$

which is well-defined unless $\cos(x) = 0$. So, the function is differentiable for all real values x except for $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$.

Marking scheme:

- 1 mark for writing that the function is differentiable whenever the denominator is nonzero.
- 1 mark for solving correctly and finding **all** points where the function is not differentiable.

Long answer question — you must show your work

3. 4 marks If $\sin(y)\sqrt{x+6} + x^2 + x\cos(y) = 6$, then find y' at the points where $y = \pi$. You must justify your answer.

Solution:

- First we find the x-coordinates where $y = \pi$.

$$\begin{aligned} \sqrt{x+6}\sin(\pi) + x^2 + x\cos(\pi) &= 6 \\ 0 + x^2 - x - 6 &= 0 \\ (x-3)(x+2) &= 0 \end{aligned}$$

So $x = -2, 3$.

- Now we use implicit differentiation to get y' in terms of x, y:

$$\sqrt{x+6}\sin(y) + x^2 + x\cos(y) = 6$$

differentiate both sides:

$$\frac{1}{2\sqrt{x+6}}\sin(y) + \sqrt{x+6}\cos(y)y' + 2x + \cos(y) - x\sin(y)y' = 0$$

- Now set $y = \pi$ to get

$$\begin{aligned} \frac{1}{2\sqrt{x+6}}\sin(\pi) + \sqrt{x+6}\cos(\pi)y' + 2x + \cos(\pi) - x\sin(\pi)y' &= 0 \\ 0 - \sqrt{x+6}y' + 2x - 1 - 0 &= 0 \\ y' &= \frac{2x-1}{\sqrt{x+6}} \end{aligned}$$

- So at $(x, y) = (-2, \pi)$ we have $y' = \dfrac{-4-1}{\sqrt{4}} = -\dfrac{5}{2}$,
- and at $(x, y) = (3, \pi)$ we have $y' = \dfrac{6-1}{\sqrt{9}} = \dfrac{5}{3}$.

Marking scheme:

- 1 mark for finding BOTH $x = -2, 3$ as the x-values when $y = \pi$ on the given curve.
- 1 mark for implicit differentiation (they do not need to isolate y').
- Note that students can get one of the above without the other. ie if students implicit-diff but not work out x, then they get 1 mark. Similarly if students get both x but do not implicit-diff, they get 1 mark.
- 1 mark for computing $y' = -5/2$ at $x = -2$.
- 1 mark for computing $y' = 5/3$ at $x = 3$.