First Name: ____________ Last Name: ____________

Student-No: ____________ Section: ____________

Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

   Consider the function, $h(x) = 2x^3 - 6x^2 + 2$.

   (a) What are the coordinates of the **local** maximum of $h(x)$?

   Answer: $(0, 2)$

   Solution: The function $h(x)$ has critical points, but no singular points since its derivative $h'(x) = 6x^2 - 12x$ is defined for all values of x. The critical points of $h(x)$ are computed by equating $h(x) = 0$ which yields

   $$6x^2 - 12x = 0, \text{ i.e. } x(x-2) = 0,$$

   and thus $x = 0$ and $x = 2$. Using either the Second Derivative Test, i.e. computing $h''(x) = 12x - 12$ and then plugging in the critical numbers $x = 0$ and $x = 2$ in $h''(x)$, or by simply noticing that $h'(x)$ changes from positive to negative at $x = 0$ and from negative to positive at $x = 2$, we conclude that $x = 0$ is a point of local maximum, while $x = 2$ is a point of local minimum. We compute $f(0) = 2$ and $f(2) = 2 \times 8 - 6 \times 4 + 2 = -6$.

   (b) What are the coordinates of the **local** minimum of $h(x)$?

   Answer: $(2, -6)$

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

   (a) Find the intervals where $f(x) = \arcsin(x) + 2\sqrt{1-x^2}$ is increasing.

   Solution: First of all, $f(x)$ is only defined on $[-1, 1]$. In order to find where $f(x)$ is increasing we compute $f'(x)$ and see where it is positive.

   $$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^2}} = \frac{1-2x}{\sqrt{1-x^2}}$$

   This is defined on $(-1, 1)$. The denominator is positive on that domain, while the numerator is positive when $1 - 2x > 0$. Hence f is increasing on $(-1, 1/2)$.

   Marking scheme:

   - 1 mark for computing $f'(x)$, i.e. $f'(x) = \frac{1-2x}{\sqrt{1-x^2}}$.
   - 1 mark for writing the correct interval where $f(x)$ is increasing; also accept interval including one or both endpoints.

(b) Let $f(x) = (x-\pi)^2 - \sin(x) + \cos(x)$. Show that there exists a real number c such that $f'(c) = 0$.

Solution: We note that $f(0) = f(2\pi) = \pi^2 - 0 + 1$. Then using the Mean Value Theorem (note that the function is differentiable for all real numbers), we get that there exists $c \in (0, 2\pi)$ such that

$$f'(c) = \frac{f(2\pi) - f(0)}{2\pi - 0} = 0.$$

Marking scheme:

- 1 mark for writing that $f(0) = f(2\pi) = 1 + \pi^2$.
- 1 mark for invoking Mean Value Theorem (or Rolle) correctly to conclude the existence of $c \in (0, 2\pi)$ such that $f'(c) = 0$.
- **Alternatively** the students could differentiate f and then apply Intermediate Value Theorem (IVT) for $f'(x)$. Since $f'(x) = 2(x-\pi) - \cos x - \sin x$, one notes that $f(0) = -2\pi - 1 < 0$ and $f(2\pi) = 2\pi - 1 > 0$ and since $f'(x)$ is continuous for all real values, then there exists $c \in (0, 2\pi)$ such that $f'(c) = 0$, by IVT. For this approach = 1 mark for evaluating derivative at 2 sensible points and 1 mark for applying IVT.

Long answer question — you must show your work

3. 4 marks Find the global maximum and the global minimum for $f(x) = 3x^4 - 4x^3 + 3$ on the interval $[-1, 2]$.

Solution: We compute $f'(x) = 12x^3 - 12x^2$, which means that $f(x)$ has no singular points (i.e., it is differentiable for all values of x), but it has two critical points obtained by solving $f'(x) = 0$, i.e. $0 = 12(x^3 - x^2) = 4x^2(x-1)$ which yields the two critical points $x = 0$ and $x = 1$. In order to compute the global maximum and the global minimum for $f(x)$ on the interval $[-1, 2]$, we compute

$$f(-1) = 3 + 4 + 3 = 10 \qquad f(2) = 48 - 32 + 3 = 19$$
$$f(0) = 3 \qquad f(1) = 2$$

So, the global maximum is $f(2) = 19$ while the global minimum is $f(1) = 2$.

Marking scheme:

- 1 mark for finding $x = 0, 1$ as the only critical (and singular) points in $[-1, 2]$.
- 2 marks for computing $f(-1), f(2), f(0)$ and $f(1)$. If they compute one value wrong (or they are not computed), then they lose 1 mark. If at least two values values are computed wrong (or not computed), then they lose 2 marks.
- 1 mark for writing correctly that $f(1) = 2$ is the global minimum, while $f(2) = 19$ is the global maximum.

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Very short answer questions

1. [2 marks] Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

   Consider the function, $h(x) = x^3 - 3x + 5$.

   (a) What are the coordinates of the **local** maximum of $h(x)$?

   Answer: $(-1, 7)$

   Solution: The function $h(x)$ has critical points, but no singular points since its derivative $h'(x) = 3x^2 - 3$ is defined for all values of x. The critical points of $h(x)$ are computed by equating $h(x) = 0$ which yields

   $$3x^2 - 3 = 0, \text{ i.e. } x^2 = 1,$$

   and thus $x = -1$ and $x = 1$. Using either the Second Derivative Test, i.e. computing $h''(x) = 6x$ and then plugging in the critical numbers $x = -$ and $x = 1$ in $h''(x)$, or by simply noticing that $h'(x)$ changes from positive to negative at $x = -1$ and from negative to positive at $x = 1$, we conclude that $x = -1$ is a point of local maximum, while $x = 1$ is a point of local minimum. We compute $f(-1) = 7$ and $f(1) = 3$.

   (b) What are the coordinates of the **local** minimum of $h(x)$?

   Answer: $(1, 3)$

Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.

   (a) Find the intervals where $f(x) = \frac{\sqrt{x}}{x+6}$ is increasing.

   Solution: First of all, $f(x)$ is defined for all $x \geq 0$ due to the presence of the square-root. In order to find where is $f(x)$ increasing, we find where is $f'(x)$ positive. So,

   $$f'(x) = \frac{\frac{1}{2\sqrt{x}} \cdot (x+6) - \sqrt{x} \cdot 1}{(x+6)^2} = \frac{x + 6 - 2x}{2\sqrt{x} \cdot (x+6)^2} = \frac{6 - x}{2\sqrt{x} \cdot (x+6)^2}$$

   and thus, since the denominator is always positive, we conclude that $f(x)$ is increasing when $f'(x) > 0$, i.e. when $6 - x > 0$. Recalling that the domain of definition for $f(x)$ is $[0, +\infty)$, we conclude that $f(x)$ is increasing on the interval $(0, 6)$. **Marking scheme:**

   - 1 mark for computing $f'(x)$, i.e. $f'(x) = \frac{6-x}{2\sqrt{x} \cdot (x+6)^2}$.
   - 1 mark for writing the correct interval where $f(x)$ is increasing; it is acceptable also $[0, 6]$, but **not** acceptable $(-\infty, 6)$.

(b) Let $f(x) = x^2 - 2\pi x - \sin(x)$. Show that there exists a real number c such that $f'(c) = 0$.

Solution: We note that $f(0) = f(2\pi) = 0$. Then using the Mean Value Theorem (note that the function is differentiable for all real numbers), we get that there exists $c \in (0, 2\pi)$ such that

$$f'(c) = \frac{f(2\pi) - f(0)}{2\pi - 0} = 0.$$

Marking scheme:

- 1 mark for writing that $f(0) = f(2\pi) = 0$.
- 1 mark for invoking Mean Value Theorem (or Rolle) correctly to conclude the existence of $c \in (0, 2\pi)$ such that $f'(c) = 0$.
- **Alternatively** the students could differentiate f and then apply Intermediate Value Theorem (IVT) for $f'(x)$. Since $f'(x) = 2x - 2\pi - \cos(x)$, one notes that $f'(0) = -2\pi - 1 < 0$ and $f'(2\pi) = 2\pi - 1 > 0$ and since $f'(x)$ is continuous for all real values, then there exists $c \in (0, 2\pi)$ such that $f'(c) = 0$, by IVT. For this approach = 1 mark for evaluating derivative at 2 sensible points and 1 mark for applying IVT.

Long answer question — you must show your work

3. [4 marks] Find the global maximum and the global minimum for $f(x) = x^3 - 6x^2 + 2$ on the interval $[3, 5]$.

Solution: We compute $f'(x) = 3x^2 - 12x$, which means that $f(x)$ has no singular points (i.e., it is differentiable for all values of x), but it has two critical points obtained by solving $f'(x) = 0$, i.e. $3x(x - 4) = 0$ which yields the two critical points $x = 0$ and $x = 4$. In order to compute the global maximum and the global minimum for $f(x)$ on the interval $[3, 5]$, we compute

$$f(3) = -25, \ f(4) = -30 \text{ and } f(5) = -23.$$

So, the global maximum is $f(5) = -23$ while the global minimum is $f(4) = -30$.

Marking scheme:

- 1 mark for writing that $x = 4$ is the only critical (and singular) point of $f(x)$ in the interval $[3, 5]$.
- 2 marks for computing $f(3)$, $f(4)$ and $f(5)$. If they compute one or two values wrong (or they are not computed), then they lose 1 mark. If all three values are computed wrong (or not computed), then they lose 2 marks.
- 1 mark for stating BOTH $f(4) = -30$ is the global minimum, and $f(5) = -23$ is the global maximum.

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

Consider the function, $h(x) = x^3 - 12x + 4$.

(a) What are the coordinates of the **local** maximum of $h(x)$?

Answer: $(-2, 20)$

Solution: The function $h(x)$ has critical points, but no singular points since its derivative $h'(x) = 3x^2 - 12$ is defined for all values of x. The critical points of $h(x)$ are computed by equating $h(x) = 0$ which yields

$$3x^2 - 12 = 0, \text{ i.e. } x^2 = 4,$$

and thus $x = -2$ and $x = 2$. Using either the Second Derivative Test, i.e. computing $h''(x) = 6x$ and then plugging in the critical numbers $x = -2$ and $x = 2$ in $h''(x)$, or by simply noticing that $h'(x)$ changes from positive to negative at $x = -2$ and from negative to positive at $x = 2$, we conclude that $x = -2$ is a point of local maximum, while $x = 2$ is a point of local minimum. We compute $f(-2) = 20$ and $f(2) = -12$.

(b) What are the coordinates of the **local** minimum of $h(x)$?

Answer: $(2, -12)$

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

(a) Find the intervals where $f(x) = \frac{\sqrt{x-1}}{2x+4}$ is increasing.

Solution: First of all, $f(x)$ is defined for all $x \geq 1$ due to the presence of the square-root. We also need $x \neq -2$ for the denominator, but this is covered by $x \geq 1$. In order to find where is $f(x)$ increasing, we find where is $f'(x)$ positive. So,

$$f'(x) = \frac{\frac{2x+4}{2\sqrt{x-1}} - 2\sqrt{x-1}}{(2x+4)^2} = \frac{(x+2) - 2(x-1)}{\sqrt{x-1}(2x+4)^2} = \frac{-x+4}{\sqrt{x-1}(2x+4)^2}$$

Note the denominator is never negative, so we conclude that $f(x)$ is increasing when the numerator of $f'(x)$ is positive, i.e. when $4 - x > 0$, or $x < 4$. Recalling that the domain of definition for $f(x)$ is $[1, +\infty)$, we conclude that $f(x)$ is increasing on the interval $(1, 4)$. **Marking scheme:**

- 1 mark for computing $f'(x)$, i.e. $f'(x) = \frac{4-x}{\sqrt{x-1}(2x+4)^2}$.
- 1 mark for writing the correct interval where $f(x)$ is increasing; it is acceptable also $[1, 4]$, but **not** acceptable $(-\infty, 4)$.

(b) Let $f(x) = x^2 - 3\pi x + \sin(x)$. Show that there exists a real number c such that $f'(c) = 0$.

Solution: We note that $f(0) = f(3\pi) = 0$. Then using the Mean Value Theorem (note that the function is differentiable for all real numbers), we get that there exists $c \in (0, 3\pi)$ such that

$$f'(c) = \frac{f(3\pi) - f(0)}{3\pi - 0} = 0.$$

Marking scheme:

- 1 mark for writing that $f(0) = f(3\pi) = 0$.
- 1 mark for invoking Mean Value Theorem (or Rolle) correctly to conclude the existence of $c \in (0, 3\pi)$ such that $f'(c) = 0$.
- **Alternatively** the students could differentiate f and then apply Intermediate Value Theorem (IVT) for $f'(x)$. Since $f'(x) = 2x - 3\pi + \cos(x)$, one notes that $f'(0) = -3\pi + 1 < 0$ and $f'(3\pi) = 3\pi - 1 > 0$ and since $f'(x)$ is continuous for all real values, then there exists $c \in (0, 3\pi)$ such that $f'(c) = 0$, by IVT. For this approach = 1 mark for evaluating derivative at 2 sensible points and 1 mark for applying IVT.

Long answer question — you must show your work

3. [4 marks] Find the global maximum and the global minimum for $f(x) = x^5 - 5x - 10$ on the interval $[0, 2]$.

Solution: We compute $f'(x) = 5x^4 - 5$, which means that $f(x)$ has no singular points (i.e., it is differentiable for all values of x), but it has two critical points obtained by solving $f'(x) = 0$, i.e. $5(x^4 - 1) = 0$ which yields the two critical points $x = -1$ and $x = 1$. In order to compute the global maximum and the global minimum for $f(x)$ on the interval $[0, 2]$, we compute

$$f(0) = -10, \ f(1) = -14 \text{ and } f(2) = 12.$$

So, the global maximum is $f(2) = 12$ while the global minimum is $f(1) = -14$.

Marking scheme:

- 1 mark for writing that $x = 1$ is the only critical (and singular) point of $f(x)$ in the interval $[0, 2]$.
- 2 marks for computing $f(0)$, $f(1)$ and $f(2)$. If they compute one or two values wrong (or they are not computed), then they lose 1 mark. If all three values are computed wrong (or not computed), then they lose 2 marks.
- 1 mark for stating BOTH $f(1) = -14$ is the global minimum, and $f(2) = 12$ is the global maximum.

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

Consider the function, $h(x) = 2x^3 - 6x + 2$.

(a) What are the coordinates of the **local** maximum of $h(x)$?

Answer: $(-1, 6)$

Solution: The function $h(x)$ has critical points, but no singular points since its derivative $h'(x) = 6x^2 - 6$ is defined for all values of x. The critical points of $h(x)$ are computed by equating $h(x) = 0$ which yields

$$6x^2 - 6 = 0, \text{ i.e. } x^2 - 1 = 0,$$

and thus $x = -1$ and $x = +1$. Using either the Second Derivative Test, i.e. computing $h''(x) = 6x$ and then plugging in the critical numbers $x = -1$ and $x = +1$ in $h''(x)$, or by simply noticing that $h'(x)$ changes from positive to negative at $x = -1$ and from negative to positive at $x = +1$, we conclude that $x = -1$ is a point of local maximum, while $x = +1$ is a point of local minimum. We compute $f(1) = -2$ and $f(-1) = -2 + 6 + 2 = 6$.

(b) What are the coordinates of the **local** minimum of $h(x)$?

Answer: $(1, -2)$

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

(a) Find the intervals where $f(x) = xe^{-x^2/2}$ is increasing.

Solution: The function is defined on all reals. Its derivative is

$$f'(x) = e^{-x^2/2} + x \cdot (-x) \cdot e^{-x^2/2} = (1 - x^2)e^{-x^2/2}$$

Since $e^{blah} > 0$, the sign of the derivative is determined by the sign of $(1 - x^2)$. Hence the function is increasing when $(1 - x^2) > 0$, that is when $x \in (-1, 1)$.

Marking scheme:

- 1 mark for computing $f'(x)$, i.e. $f'(x) = (1 - x^2)e^{-x^2/2}$.
- 1 mark for writing the correct interval where $f(x)$ is increasing; also accept interval with one or both endpoints.

(b) Let $f(x) = (x + \pi)^2 + \cos(x)$. Show that there exists a real number c such that $f'(c) = 0$.

Solution: We note that $f(0) = f(-2\pi) = \pi^2 + 1$. Then using the Mean Value Theorem (note that the function is differentiable for all real numbers), we get that there exists $c \in (-2\pi, 0)$ such that

$$f'(c) = \frac{f(-2\pi) - f(0)}{-2\pi} = 0.$$

Marking scheme:

- 1 mark for writing that $f(0) = f(-2\pi) = 1 + \pi^2$.
- 1 mark for invoking Mean Value Theorem (or Rolle) correctly to conclude the existence of $c \in (-2\pi, 0)$ such that $f'(c) = 0$.
- **Alternatively** the students could differentiate f and then apply Intermediate Value Theorem (IVT) for $f'(x)$. Since $f'(x) = 2(x + \pi) - \sin x$, one notes that $f(0) = 2\pi > 0$ and $f(-2\pi) = -2\pi < 0$ and since $f'(x)$ is continuous for all real values, then there exists $c \in (-2\pi, 0)$ such that $f'(c) = 0$, by IVT. For this approach = 1 mark for evaluating derivative at 2 sensible points and 1 mark for applying IVT.

Long answer question — you must show your work

3. 4 marks Find the global maximum and the global minimum for $f(x) = 4x^3 - 6x^2 + 3$ on the interval $[-1, 2]$.

Solution: We compute $f'(x) = 12x^2 - 12x$, which means that $f(x)$ has no singular points (i.e., it is differentiable for all values of x), but it has two critical points obtained by solving $f'(x) = 0$, i.e. $0 = 12x(x - 1)$ which yields the two critical points $x = 0$ and $x = 1$. In order to compute the global maximum and the global minimum for $f(x)$ on the interval $[-1, 2]$, we compute

$$f(-1) = -4 - 6 + 3 = -7 \qquad f(2) = 32 - 24 + 3 = 11$$
$$f(0) = 3 \qquad f(1) = 1$$

So, the global maximum is $f(2) = 11$ while the global minimum is $f(-1) = -7$.

Marking scheme:

- 1 mark for finding $x = 0, 1$ as the only critical (and singular) points in $[-1, 2]$.
- 2 marks for computing $f(-1), f(2), f(0)$ and $f(1)$. If they compute one value wrong (or they are not computed), then they lose 1 mark. If at least two values values are computed wrong (or not computed), then they lose 2 marks.
- 1 mark for writing correctly that $f(2) = 11$ is the global max, while $f(-1) = -7$ is the global maximum.