First Name: ____________ Last Name: ____________

Student-No: ____________ Section: ____________

Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

   Consider a function, $h(x)$, whose third-degree Maclaurin polynomial is $1 - 3x + \frac{1}{6}x^2 + \frac{2}{7}x^3$.

   (a) What is $h'(0)$?

   Answer: -3

   Solution: The third Maclaurin polynomial for $h(x)$ is

   $$h(0) + h'(0)x + \frac{h''(0)}{2} \cdot x^2 + \frac{h'''(0)}{6} \cdot x^3 = 1 - 3x + \frac{1}{6}x^2 + \frac{2}{7}x^3$$

   Thus $h'(0) = -3$ and for the next part, we note that $h''(0) = \frac{1}{3}$.

   (b) What is $h''(0)$?

   Answer: $1/3$

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

   (a) Estimate $\sqrt[4]{15}$ using a linear approximation

   Solution: We use the function $f(x) = x^{1/4}$ and point $a = 15$ as the centre of our approximation since we know that

   $$f(a) = f(16) = \sqrt[4]{16} = 2.$$

   We compute $f'(x) = \frac{1}{4}x^{-3/4}$; so

   $$f'(16) = \frac{1}{4 \cdot 8} = \frac{1}{32}.$$

   So, a linear approximation of $\sqrt[4]{15} = f(15)$ is

   $$T_1(15) = f(16) + f'(16) \cdot (15 - 16) = 2 - \frac{1}{32} = \frac{63}{32}$$

   Marking scheme:

   - 1 mark for writing that the function to use for the linear approximation is $f(x) = \sqrt{x}$, that $a = 16$. Also accept nearby a values that make it easy to compute $f(a), f'(a)$.
   - 1 mark for obtaining the linear approximation $63/32$. (or equivalent if they use other acceptable a)

(b) Consider a function $f(x)$ which has $f^{(3)}(x) = \frac{x}{10 - \sin x}$. Show that when we approximate $f(1)$ using its second degree Maclaurin polynomial, the absolute value of the error is less than $\frac{1}{50} = 0.02$.

Solution:

- The error is bounded (in absolute value) by

$$\max_{c\in[0,1]} \left|\frac{f'''(c)}{3!} \cdot (1-0)^3\right| = \max_{c\in[0,1]} \left|\frac{c}{6(10-\sin c)}\right|.$$

- When $0 \le c \le 1$, we know that $-1 \le \sin c \le 1$. Hence $9 \le 10 - \sin c \le 11$.
- Hence

$$\begin{aligned}\left|\frac{c}{6(10-\sin c)}\right| &= \frac{|c|}{6|10-\sin c|}\\ &\le \frac{1}{6|10-\sin c|}\\ &\le \frac{1}{6\times 9} = \frac{1}{54} < \frac{1}{50}\end{aligned}$$

Marking scheme:

- 1 mark for writing that the error is bounded (in absolute value) by

$$\max_{c\in[0,1]} \left|\frac{f'''(c)}{3!} \cdot (1-0)^3\right| = \max_{c\in[0,1]} \left|\frac{c}{10-\sin c}\right|.$$

  (or similar statement)
- 1 mark for explaining their error bound (provided the error is still bounded above by 0.02). Be reasonably generous.
- **The students lose 1 mark if they get the right bound, but don't explain it.**

Long answer question — you must show your work

3. [4 marks] Two particles move in the cartesian plane. Particle A starts at $(3, 0)$ while particle B starts at $(0, 0)$. Particle A moves in the $+y$ direction at 1 unit per second, while B moves in the $-y$ direction at 3 units per second. How fast is the distance between the particles changing when the distance between them is 5 units?

Solution:

- Let y_1 be the distance travelled by A and y_2 be the distance travelled by B.

- The distance between them is then

$$z^2 = 9 + (y_1 + y_2)^2$$

- We differentiate the above equation with respect to t and get

$$2z \cdot z' = 2(y_1 + y_2) \cdot (y_1' + y_2')$$

- We need to determine y_1, y_2 when $z = 5$. We have

$$5^2 = 9 + (y_1 + y_2)^2$$

Hence $y_1 + y_2 = 4$. So we must have $y_1 = 1, y_2 = 3$.

- We are told that $y_1' = 1$ and $y_2' = 3$. Hence

$$2 \times 5 \times z' = 2 \times 4 \times 4$$
$$z' = \frac{16}{5} \text{units / second}$$

- Equivalently we can let y be the vertical distance between the particles. Hence $y' = 4$ and

$$z^2 = 9 + y^2$$
$$2zz' = 2yy'$$

When $z = 5, y = 4$ and so

$$10z' = 32$$

Thus $z' = 16/5$ units / second.

Marking scheme:

- 1 mark for $z^2 = 9 + (y_1 + y_2)^2$ or equivalently $z = 9 + y^2$.
- 1 mark for obtaining the equation $2z(t) \cdot z'(t) = 2(y_1 + y_2)(y_1' + y_2')$ or equivalently $2zz' = 2yy'$.
- 1 mark for $y_1 = 1, y_2 = 3$ all correct (equivalently $y = 4$) when $z = 5$.
- 1 mark for obtaining the correct answer $z' = \frac{16}{5}$.

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Student-No: ______ Section: ______

Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

   Consider a function, $h(x)$, whose third-degree Maclaurin polynomial is $5 - \frac{1}{3}x^2 + 2x^3$.

   (a) What is $h'(0)$?

   Answer: 0

   Solution: The third Maclaurin polynomial for $h(x)$ is

   $$h(0) + h'(0)x + \frac{h''(0)}{2} \cdot x^2 + \frac{h'''(0)}{6} \cdot x^3 = 5 - \frac{x^2}{3} + 2x^3.$$

   Thus $h'(0) = 0$ and for the next part, we note that $h''(0) = -\frac{2}{3}$.

   (b) What is $h''(0)$?

   Answer: $-\frac{2}{3}$

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

   (a) Estimate $\sqrt{35}$ using a linear approximation

   Solution: We use the function $f(x) = \sqrt{x}$ and point $a = 36$ as the centre of our approximation since we know that

   $$f(a) = f(36) = \sqrt{36} = 6.$$

   We compute $f'(x) = \frac{1}{2\sqrt{x}}$; so

   $$f'(36) = \frac{1}{2\sqrt{36}} = \frac{1}{12}.$$

   So, a linear approximation of $\sqrt{35} = f(35)$ is

   $$T_1(35) = f(36) + f'(36) \cdot (35 - 36) = 6 - \frac{1}{12} = \frac{71}{12}.$$

   Marking scheme:

   - 1 mark for writing that the function to use for the linear approximation is $f(x) = \sqrt{x}$, that $a = 36$. Also accept nearby a values that make it easy to compute $f(a), f'(a)$ (eg 25,49).
   - 1 mark for obtaining the linear approximation $6 - \frac{1}{12}$. (or equivalent if they use other acceptable a)

(b) Consider a function $f(x)$ which has $f^{(3)}(x) = \dfrac{x^3}{10-x^2}$. Show that when we approximate $f(1)$ using its second degree Maclaurin polynomial, the absolute value of the error is less than $\frac{1}{50} = 0.02$.

Solution:

- The error is bounded (in absolute value) by

$$\max_{c\in[0,1]} \left| \frac{f'''(c)}{3!} \cdot (1-0)^3 \right| = \max_{c\in[0,1]} \left| \frac{c^3}{6(10-c^2)} \right|.$$

- Since $c \in [0,1]$, we know that $\left|\frac{c^3}{6(10-c^2)}\right| = \frac{c^3}{6(10-c^2)}$ since both numerator and denominator are positive.

- When $0 \le c \le 1$, we know that $0 \le c^3 \le 1$ and $9 \le 10 - c^2 \le 10$, and that numerator and denominator are non-negative, so

$$\begin{aligned} \left| \frac{c^3}{6(10-c^2)} \right| &= \frac{c^3}{6(10-c^2)} \le \frac{1}{6(10-c^2)} \le \frac{1}{6\cdot 9} \\ &= \frac{1}{54} \le \frac{1}{50} \end{aligned}$$

  as required.

- Alternatively, notice that c^3 is an increasing function of c, while $10 - c^2$ is a decreasing function of c. Hence the fraction is an increasing function of c and takes its largest value at $c = 1$. Hence

$$\left| \frac{c^3}{6(10-c^2)} \right| \le \frac{1}{6\times 9} = \frac{1}{54} \le \frac{1}{50}.$$

Marking scheme:

- 1 mark for writing that the error is bounded (in absolute value) by

$$\max_{c\in[0,1]} \left| \frac{f'''(c)}{3!} \cdot (1-0)^3 \right| = \max_{c\in[0,1]} \left| \frac{c^3}{6(10-c^2)} \right|.$$

  (or similar statement)

- 1 mark for explaining why $c = 1$ is the right choice and then verifying that in that case the error is still bounded above by 0.02. Be reasonably generous.

- **The students lose 1 mark if they don't explain why $c = 1$ is the right choice (and simply they plug in $c = 1$, or compare the values they get between plugging in $c = 0$ and $c = 1$).**

Long answer question — you must show your work

3. 4 marks Two particles move in the cartesian plane. Particle A travels on the x-axis starting at $(10, 0)$ and moving towards the origin with a speed of 2 units per second. Particle B travels on the y-axis starting at $(0, 12)$ and moving towards the origin with a speed of 3 units per second. What is the rate of change of the distance between the two particles when particle A reaches the point $(4, 0)$?

Solution:

- We compute the distance $z(t)$ between the particle and the point with coordinates $(0, 3)$ at each moment in time as

$$z^2(t) = x(t)^2 + y(t)^2,$$

where $x(t)$ is the position on the x-axis of the particle A at time t (measured in seconds) and $y(t)$ is the position on the y-axis of the particle B at the same time t.

- We differentiate the above equation with respect to t and get

$$2z \cdot z' = 2x \cdot x' + 2y \cdot y',$$

- We are told that $x' = -2$ and $y' = -3$. Further it will take 3 seconds for particle A to reach $x = 4$, and in this time particle B will reach $y = 3$.
- Alternatively (equivalently??) write $x = 10 - 2t, y = 12 - 3t$ to get $t = 3, x' = -2, y' = -3, y = 3$.
- At this point $z = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = 5$.
- Hence

$$10z' = 8 \cdot (-2) + 6 \cdot (-3) = -34$$
$$z' = -\frac{34}{10} = -\frac{17}{5} \text{ units per second.}$$

Marking scheme:

- 1 mark for obtaining the equation $2z(t) \cdot z'(t) = 2x \cdot x' + 2y \cdot y'$.
- 1 mark for $x' = -2, y' = -3, y = 3$ all correct.
- 1 mark for computing $z(3) = 5$.
- 1 mark for obtaining the correct answer $z'(3) = -\frac{17}{5}$.

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Student-No: ______________________ Section: ______________________

Very short answer questions

1. [2 marks] Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

   Consider a function, $f(x)$, whose third-degree Maclaurin polynomial is $4 + 3x^2 + \frac{1}{2}x^3$.

   (a) What is $f'(0)$?

   Answer: 0

   Solution: The third Maclaurin polynomial for $f(x)$ is

   $$f(0) + f'(0)x + \frac{f''(0)}{2} \cdot x^2 + \frac{f'''(0)}{6} \cdot x^3 = 4 + 3x^2 + \frac{1}{2}x^3.$$

   Thus $f'(0) = 0$ and for the next part, we note that $f''(0) = 6$.

   (b) What is $f''(0)$?

   Answer: 6

Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.

   (a) Estimate $\sqrt[3]{9}$ using a linear approximation

   Solution: We use the function $f(x) = \sqrt[3]{x}$ and point $a = 8$ as the centre of our approximation since we know that

   $$f(a) = f(8) = \sqrt[3]{8} = 2.$$

   We compute $f'(x) = \frac{1}{3\sqrt[3]{x}^2}$; so

   $$f'(8) = \frac{1}{3\sqrt[3]{8}^2} = \frac{1}{12}.$$

   So, a linear approximation of $\sqrt[3]{9} = f(9)$ is

   $$T_1(9) = f(8) + f'(8) \cdot (9 - 8) = 2 + \frac{1}{12}.$$

   Marking scheme:

   - 1 mark for writing that the function to use for the linear approximation is $f(x) = \sqrt[3]{x}$, that $a = 8$. Also accept nearby a values that make it easy to compute $f(a), f'(a)$.
   - 1 mark for obtaining the linear approximation $6 + \frac{1}{12}$. (or equivalent if they use other acceptable a)

(b) Consider a function $f(x)$ which has $f^{(3)}(x) = \frac{1}{5}e^{-2x} \cdot \sin(x)$. Show that when we approximate $f(1)$ using its second degree Maclaurin polynomial, the absolute value of the error is less than $\frac{1}{30}$.

Solution:

- The error is bounded (in absolute value) by

$$\max_{c\in[0,1]} \left| \frac{f'''(c)}{3!} \cdot (1-0)^3 \right| = \max_{c\in[0,1]} \left| \frac{e^{-2c}\sin(c)}{6(5)} \right|.$$

- Since $c \in [0,1]$, we know that $1 = e^0 \geq e^{-2c} \geq e^{-2}$, and $-1 \leq \sin c \leq 1$. Hence

$$\begin{aligned} \left| \frac{e^{-2c}\sin(c)}{6(5)} \right| &= \frac{1}{30} \cdot |e^{-2c}| \cdot |\sin c| \\ &\leq \frac{1}{30} \cdot 1 \cdot 1 = \frac{1}{30} \end{aligned}$$

as required.

Marking scheme:

- 1 mark for writing that the error is bounded (in absolute value) by

$$\max_{c\in[0,1]} \left| \frac{f'''(c)}{3!} \cdot (1-0)^3 \right| = \max_{c\in[0,1]} \left| \frac{c^3}{6(10-c^2)} \right|.$$

(or similar statement)

- 1 mark for explaining the bound and verifying that the error is still bounded above by $1/30$. Be reasonably generous.
- **The students lose 1 mark if they don't have some explanation of the bound**

Long answer question — you must show your work

3. 4 marks Two particles move in the Cartesian plane. Particle A starts at $(2,0)$ and moves on the x-axis away from the origin at 1 unit per second. Particle B starts at the origin, and moves along the y-axis at 2 units per second (in the $+y$-direction). How fast is the distance between the particles increasing when A reaches $(6,0)$?

Solution:

- Let A's position by $(a,0)$ and B's position be $(0,b)$. Then we know $\frac{da}{dt} = 1$ unit per second, and $\frac{db}{dt} = 2$ units per second. Note these rates are both positive.

- Our units are measured in seconds. Let the start time be $t = 0$. If z is the distance between the particles, we want to know $\frac{dz}{dt}$ when $a = 6$.
- So, we need an equation relating a, b, and z. Of course this equation is

$$z^2 = a^2 + b^2$$

and we differentiate with respect to t:

$$2zz' = 2aa' + 2bb'$$

- To solve for $\frac{dz}{dt}$, we need a and z, when $a = 6$. It will take 4 seconds for A to reach this point, and in that time B moves 8 units. Hence $a = 6$, $b = 8$ and so

$$z^2 = 6^2 + 8^2 = 100$$

thus $z = 10$.

- Now we solve for $\frac{dz}{dt}$:

$$\begin{aligned} 2zz' &= 2aa' + 2bb' \\ 20z' &= 12 \cdot 1 + 2 \cdot 8 \cdot 2 = 12 + 32 = 44 \\ z &= \boxed{\frac{11}{5}} \text{ units per second} \end{aligned}$$

- Equivalently, we can write $a = 2 + t$ and $b = 2t$, so $z^2 = a^2 + b^2 = (2+t)^2 + (2t)^2$. Then

$$2z\frac{dz}{dt} = 2(2+t) + 8t$$

So

$$2(10)\left.\frac{dz}{dt}\right|_{t=4} = 2(2+4) + 8(4) = 44$$

hence $\frac{dz}{dt} = \boxed{\frac{11}{5}}$ units per second.

Marking scheme:

- 1 mark for obtaining the equation $2z(t) \cdot z'(t) = 2a \cdot a' + 2b \cdot b'$.
- 1 mark for $a' = 1, b' = 2, b = 8$ all correct.
- 1 mark for computing $z(4) = 10$.
- 1 mark for obtaining the correct answer $z'(4) = \frac{11}{5}$.

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Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

   Consider a function, $h(x)$, whose third-degree Maclaurin polynomial is $1 + 4x - \frac{1}{3}x^2 + \frac{3}{4}x^3$.

   (a) What is $h^{(3)}(0)$?

   Answer: 9/2

   Solution: The third Maclaurin polynomial for $h(x)$ is

   $$h(0) + h'(0)x + \frac{h''(0)}{2} \cdot x^2 + \frac{h^{(3)}(0)}{6} \cdot x^3 = 1 + 4x - \frac{1}{3}x^2 + \frac{3}{4}x^3$$

   Thus $h^{(3)}(0) = 6 \cdot \frac{2}{3} = 4$ and for the next part, we note that $h(0) = 0$.

   (b) What is $h''(0)$?

   Answer: $-2/3$

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

   (a) Estimate $\sqrt[3]{26}$ using a linear approximation.

   Solution: We use the function $f(x) = x^{1/3}$ and point $a = 27$ as the centre of our approximation since we know that

   $$f(27) = 3$$

   We compute $f'(x) = \frac{1}{3}x^{-2/3}$; so

   $$f'(27) = \frac{1}{3} \cdot (27)^{-2/3} = \frac{1}{27}$$

   So, the linear approximation of $26^{1/3} = f(26)$ is

   $$T_1(26) = f(27) + f'(27) \cdot (26 - 27) = 3 - \frac{1}{27} = \frac{80}{27}$$

   Marking scheme:

   - 1 mark for writing that the function to use for the linear approximation is $f(x) = x^{1/3}$, that $a = 27$. Also accept nearby a values that make it easy to compute $f(a), f'(a)$.
   - 1 mark for obtaining the linear approximation 80/27. (or equivalent if they use other acceptable a)

(b) Consider a function $f(x)$ which has $f^{(3)}(x) = \dfrac{e^{-x}}{8+x^2}$. Show that when we approximate $f(1)$ using its second degree Maclaurin polynomial, the absolute value of the error is less than $1/40$.

Solution:

- The error is bounded (in absolute value) by

$$\max_{c\in[0,1]} \left| \frac{f'''(c)}{3!} \cdot (1-0)^3 \right| = \max_{c\in[-1,0]} \left| \frac{e^{-c}}{8+c^2} \right|.$$

- When $0 \le c \le 1$, we know that $1 \ge e^{-c} \ge e^{-1}$ and $8 \le 8+c^2 \le 9$, so

$$\begin{aligned} \left| \frac{e^{-c}}{6(8+c^2)} \right| &= \frac{|e^{-c}|}{6|8+c^2|} \\ &\le \frac{1}{6|8+c^2|} \\ &\le \frac{1}{6\times 8} = \frac{1}{48} < \frac{1}{40} \end{aligned}$$

as required.

- Alternatively, notice that e^{-c} is a decreasing function of c, while for $0 < c$ $8+c^2$ is an increasing function of c. Hence the fraction is a decreasing function of c and takes its largest value at $c=0$. Hence

$$\left| \frac{e^c}{6(8+c^2)} \right| \le \frac{1}{6\times 8} = \frac{1}{48} < \frac{1}{40}.$$

Marking scheme:

- 1 mark for writing that the error is bounded (in absolute value) by

$$\max_{c\in[-1,0]} \left| \frac{f'''(c)}{3!} \cdot (0-(-1))^3 \right| = \max_{c\in[-1,0]} \left| \frac{e^{-c}}{6(8+c^2)} \right|.$$

(or similar statement)

- 1 mark for explaining why $c=0$ is the right choice and then verifying that in that case the error is still bounded above by $1/48$. Be reasonably generous.
- **The students lose 1 mark if they don't explain why $c=0$ is the right choice (and they simply plug in $c=0$, or compare the values they get between plugging in $c=0$ and $c=-1$).**

Long answer question — you must show your work

Two particles move in the cartesian plane. Particle A travels on the x-axis starting at $(10,0)$ and moving towards the origin with a speed of 2 units per second. Particle B travels on the

y-axis starting at $(0, 2)$ and moving away from the origin with a speed of 4 units per second. What is the rate of change of the distance between the two particles when particle A reaches the point $(5, 0)$?

Solution:

- We compute the distance $z(t)$ between the particle and the point with coordinates $(0, 5)$ at each moment in time as

$$z^2(t) = x(t)^2 + y(t)^2,$$

  where $x(t)$ is the position on the x-axis of the particle A at time t (measured in seconds) and $y(t)$ is the position on the y-axis of the particle B at the same time t.

- We differentiate the above equation with respect to t and get

$$2z \cdot z' = 2x \cdot x' + 2y \cdot y', \qquad \text{or equivalently}$$
$$z \cdot z' = x \cdot x' + y \cdot y'.$$

- We are told that $x' = -2$ and $y' = 4$. Further it will take 2.5 seconds for particle A to reach $x = 4$, and in this time particle B will reach $y = 12$.

- Alternatively (equivalently??) write $x = 10 - 2t, y = 2 + 4t$ to get $t = 5/2, x = 5, x' = -2, y' = 4, y = 12$.

- At this point $z = \sqrt{x^2 + y^2} = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$.

- Hence

$$13z' = 5 \cdot (-2) + 13 \cdot (4) = -10 + 52 = 42$$
$$z' = \frac{42}{13} \text{ units per second.}$$

Marking scheme:

- 1 mark for obtaining the equation $2z(t) \cdot z'(t) = 2x \cdot x' + 2y \cdot y'$.
- 1 mark for $x' = -2, y' = 4, y = 12$ all correct.
- 1 mark for computing $z(2.5) = 13$.
- 1 mark for obtaining the correct answer $z'(5/2) = 42/13$.