First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your **simplified** answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

   (a) Differentiate $e^{\cos(\log x)}$. Recall that $\log x = \log_e x = \ln x$.

   Answer: $e^{\cos(\log x)} \cdot (-\sin(\log x)) \cdot \frac{1}{x}$

   Solution: This requires us to apply the chain rule twice.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\cos(\log x)}\right] &= e^{\cos(\log x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[\cos(\log x)\right] \\ &= e^{\cos(\log x)}(-\sin(\log x)) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\log x \\ &= e^{\cos(\log x)}(-\sin(\log x)) \cdot \frac{1}{x}\end{aligned}$$

   (b) If $x^2 + y^2 = \sin(x+y)$ compute $\frac{\mathrm{d}y}{\mathrm{d}x}$.

   Answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos(x+y) - 2x}{2y - \cos(x+y)}$

   Solution: Differentiate the equation

$$\begin{aligned}2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= \cos(x+y) \cdot \left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\cos(x+y) - 2x}{2y - \cos(x+y)}\end{aligned}$$

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

   (a) Differentiate $f(x) = (x^2+1)^{\sin(x)}$.

   Solution: We use logarithmic differentiation; so

$$\log f(x) = \log(x^2+1) \cdot \sin x$$

   Then differentiate to obtain

$$\frac{f'(x)}{f(x)} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\log(x^2+1) \cdot \sin x\right] = \cos x \cdot \log(x^2+1) + \frac{2x\sin x}{x^2+1}$$

In conclusion:

$$f'(x) = f(x) \cdot \left(\cos x \cdot \log(x^2+1) + \frac{2x \sin x}{x^2+1}\right) \quad \text{or equivalently}$$
$$= (x^2+1)^{\sin(x)} \cdot \left(\cos x \cdot \log(x^2+1) + \frac{2x \sin x}{x^2+1}\right)$$

Marking scheme:

- 1 mark for using logarithmic differentiation, i.e. write that $\log(f(x)) = \sin(x)^2 \cdot \ln(x^2+1)$ — provided they also write that the derivative of $\log f(x)$ is $f'(x)/f(x)$. It is not enough just to write "Use log-diff (or similar)" they have to get $f'(x)/f(x) = blah$ or $f'(x) = f(x) \times blah$.
- 1 mark for correct derivative for $f(x)$; accept as correct answer also $f'(x) = f(x) \cdot blah$.

(b) Consider a function of the form $f(x) = Ae^{kx}$ where A and k are constants. If $f(0) = 3$ and $f(2) = 5$, find the constants A and k.

Solution: The two pieces of information give us

$$f(0) = A = 3 \qquad\qquad f(2) = Ae^{2k} = 5$$

Thus we know that $A = 3$ and so $f(2) = 5 = 3e^{2k}$. Hence

$$e^{2k} = \frac{5}{3}$$
$$2k = \log(5/3)$$
$$k = \frac{1}{2} \cdot \log(5/3).$$

Marking scheme:

- One mark for each correct constant.
- This one is supposed to be straight forward.

Long answer question — you must show your work

3. 4 marks If $x^2 \cos(y) + 2xe^y = 8$, then find y' at the points where $y = 0$. You must justify your answer.

Solution:

- First we find the x-coordinates where $y = 0$.

$$\begin{aligned} x^2\cos(0) + 2xe^0 &= 8 \\ x^2 + 2x - 8 &= 0 \\ (x+4)(x-2) &= 0 \end{aligned}$$

  So $x = 2, -4$.

- Now we use implicit differentiation to get y' in terms of x, y:

$$\begin{aligned} x^2\cos(y) + 2xe^y &= 8 \qquad \text{differentiate both sides} \\ x^2 \cdot (-\sin y) \cdot y' + 2x\cos y + 2xe^y \cdot y' + 2e^y &= 0 \end{aligned}$$

- Now set $y = 0$ to get

$$\begin{aligned} x^2 \cdot (-\sin 0) \cdot y' + 2x\cos 0 + 2xe^0 \cdot y' + 2e^0 &= 0 \\ 0 + 2x + 2xy' + 2 &= 0 \\ y' = -\frac{2+2x}{2x} &= -\frac{1+x}{x} \end{aligned}$$

- So at $(x, y) = (2, 0)$ we have $y' = -\frac{3}{2}$,
- and at $(x, y) = (-4, 0)$ we have $y' = -\frac{3}{4}$.

Marking scheme:

- 1 mark for finding BOTH $x = 2, -4$ as the x-values when $y = 0$ on the given curve.
- 2 marks for implicit differentiation (they do not need to isolate y').
- Note that students can get one of the above without the other. ie if students implicit-diff but not work out x, then they get 1 mark. Similarly if students get both x but do not implicit-diff, they get 1 mark.
- 1 mark for computing at least one of $y' = -3/2$ at $x = 2$, or $y' = -3/4$ at $x = -4$.

First Name: ______________ Last Name: ______________

Student-No: ______________ Section: ______________

Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your **simplified** answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

   (a) Differentiate $e^{\sin^2(x)}$

   Answer: $2\sin(x)\cdot\cos(x)\cdot e^{\sin^2(x)}$

   Solution: This requires us to apply the chain rule twice.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\sin^2(x)}\right] &= e^{\sin^2(x)}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin^2(x)\right]\\ &= e^{\sin^2(x)}(2\sin(x))\cdot\frac{\mathrm{d}}{\mathrm{d}x}\sin(x)\\ &= e^{\sin^2(x)}(2\sin(x))\cdot\cos(x)\end{aligned}$$

   (b) If $x^3 + y^4 = \cos(x^2 + y)$ compute $\frac{\mathrm{d}y}{\mathrm{d}x}$.

   Answer: $-\dfrac{2x\sin(x^2+y)+3x^2}{4y^3+\sin(x^2+y)}$

   Solution: Differentiate the equation

$$\begin{aligned}3x^2 + 4y^3\frac{\mathrm{d}y}{\mathrm{d}x} &= -\sin(x^2+y)\cdot\left(2x+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{2x\sin(x^2+y)+3x^2}{4y^3+\sin(x^2+y)}\end{aligned}$$

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

   (a) Differentiate $f(x) = x^{\cos^3(x)}$.

   Solution: We use logarithmic differentiation; so

$$\ln f(x) = \ln(x)\cdot\cos^3(x)$$

   Then differentiate to obtain

$$\frac{f'(x)}{f(x)} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x)\cdot\cos^3(x)\right] = 3\cos^2(x)\cdot(-\sin(x))\cdot\ln(x) + \frac{\cos^3(x)}{x}$$

In conclusion:

$$f'(x) = f(x) \cdot \left(3\cos^2(x) \cdot (-\sin(x)) \cdot \ln(x) + \frac{\cos^3(x)}{x} \right) \text{ or equivalently}$$
$$= x^{\cos^3(x)} \cdot \left(-3\cos^2(x)\sin(x)\ln(x) + \frac{\cos^3(x)}{x} \right)$$

Marking scheme:

- 1 mark for using logarithmic differentiation, i.e. write that $\log(f(x)) = \ln(x) \cdot \cos^3(x)$ — provided they also write that the derivative of $\log f(x)$ is $f'(x)/f(x)$. It is not enough just to write "Use log-diff (or similar)" they have to get to $f'(x)/f(x) = blah$ or $f'(x) = f(x) \times blah$.
- 1 mark for correct derivative for $f(x)$; accept as correct answer also $f'(x) = f(x) \cdot blah$.

(b) Consider a function of the form $f(x) = Ae^{kx}$ where A and k are constants. If $f(0) = 2$ and $f(5) = 3$, find the constants A and k.

Solution: The two pieces of information give us

$$f(0) = A = 2 \qquad\qquad f(5) = Ae^{5k} = 3$$

Thus we know that $A = 2$ and so $f(5) = 3 = 2e^{5k}$. Hence

$$e^{5k} = \frac{3}{2}$$
$$5k = \ln(3/2)$$
$$k = \frac{1}{5} \cdot \ln(3/2).$$

Marking scheme:

- One mark for each correct constant.
- This one is supposed to be straight forward.

Long answer question — you must show your work

3. [4 marks] If $x^2e^y + 4x\cos(y) = 5$, then find y' at the points where $y = 0$. You must justify your answer.

Solution:

- First we find the x-coordinates where $y = 0$.

$$\begin{aligned} x^2e^0 + 4x\cos(0) &= 5 \\ x^2 + 4x - 5 &= 0 \\ (x+5)(x-1) &= 0 \end{aligned}$$

  So $x = 1, -5$.

- Now we use implicit differentiation to get y' in terms of x, y:

$$\begin{aligned} x^2e^y + 4x\cos(y) &= 5 \qquad \text{differentiate both sides} \\ x^2 \cdot e^y \cdot y' + 2xe^y + 4x(-\sin(y)) \cdot y' + 4\cos(y) &= 0 \end{aligned}$$

- Now set $y = 0$ to get

$$\begin{aligned} x^2 \cdot e^0 \cdot y' + 2xe^0 + 4x(-\sin(0)) \cdot y' + 4\cos(0) &= 0 \\ x^2y' + 2x + 4 &= 0 \\ y' &= -\frac{4+2x}{x^2}. \end{aligned}$$

- So at $(x, y) = (1, 0)$ we have $y' = -6$,
- and at $(x, y) = (-5, 0)$ we have $y' = \frac{6}{25}$.

Marking scheme:

- 1 mark for finding BOTH $x = 1, -5$ as the x-values when $y = 0$ on the given curve.
- 2 marks for implicit differentiation (they do not need to isolate y').
- Note that students can get one of the above without the other. ie if students implicit-diff but not work out x, then they get 1 mark. Similarly if students get both x but do not implicit-diff, they get 1 mark.
- 1 mark for computing at least one of $y' = -6$ at $x = 1$, or $y' = 6/25$ at $x = -5$.

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Very short answer questions

1. [2 marks] Each part is worth 1 marks. Please write your **simplified** answers in the boxes.
Marking scheme: 1 for each correct, 0 otherwise

(a) Differentiate $\sin\left(e^{5x}\right)$

Answer: $\cos\left(e^{5x}\right) \cdot e^{5x} \cdot 5$

Solution: This requires us to apply the chain rule twice.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin(e^{5x})\right] &= \cos\left(e^{5x}\right) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[e^{5x}\right] \\ &= \cos(e^{5x})(e^{5x}) \cdot \frac{\mathrm{d}}{\mathrm{d}x}[5x] \\ &= \cos(e^{5x})(e^{5x}) \cdot 5\end{aligned}$$

(b) If $e^y = xy^2 + x$, compute $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y^2+1}{e^y - 2xy}$

Solution: Differentiate the equation

$$\begin{aligned}e^y\frac{\mathrm{d}y}{\mathrm{d}x} &= x \cdot 2y\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x}\left(e^y - 2xy\right) &= y^2 + 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y^2+1}{e^y - 2xy}\end{aligned}$$

Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.

(a) Differentiate $f(x) = (x^2+1)^{(x^2+1)}$.

Solution: We use logarithmic differentiation; so

$\ln f(x) = \ln(x^2+1) \cdot (x^2+1)$

Then differentiate to obtain

$$\frac{f'(x)}{f(x)} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x^2+1) \cdot (x^2+1)\right] = \frac{2x}{x^2+1}(x^2+1) + 2x\ln(x^2+1) = 2x(1 + \ln(x^2+1))$$

In conclusion:

$$\begin{aligned} f'(x) &= f(x) \cdot 2x(1 + \ln(x^2+1)) \text{ or equivalently} \\ &= (x^2+1)^{x^2+1} \cdot 2x(1 + \ln(x^2+1)) \end{aligned}$$

Marking scheme:

- 1 mark for using logarithmic differentiation, i.e. write that $\ln(f(x)) = (x^2 + 1) \cdot \ln(x^2+1)$ — provided they also write that the derivative of $\log f(x)$ is $f'(x)/f(x)$. It is not enough just to write "Use log-diff (or similar)" they have to get to $f'(x)/f(x) = blah$ or $f'(x) = f(x) \times blah$.
- 1 mark for correct derivative for $f(x)$; accept as correct answer also $f'(x) = f(x) \cdot blah$.

(b) Consider a function of the form $f(x) = Ae^{kx}$ where A and k are constants. If $f(0) = 2$ and $f(3) = 1$, find the constants A and k.

Solution: The two pieces of information give us

$$f(0) = A = 2 \qquad\qquad f(3) = Ae^{3k} = 1$$

Thus we know that $A = 2$ and so $f(3) = 1 = 2e^{3k}$. Hence

$$\begin{aligned} e^{3k} &= \frac{1}{2} \\ 3k &= \ln(1/2) \\ k &= \frac{1}{3} \cdot \ln(1/2) = -\frac{\ln 2}{3}. \end{aligned}$$

Marking scheme:

- One mark for each correct constant. They do not need to be simplified.
- This one is supposed to be straight forward.

Long answer question — you must show your work

3. 4 marks If $x^2 + (y+1)e^y = 5$, then find y' at the points where $y = 0$. You must justify your answer.

Solution:

- First we find the x-coordinates where $y = 0$.

$$\begin{aligned} x^2 + (1)e^0 &= 5 \\ x^2 + 1 &= 5 \\ x^2 &= 4 \end{aligned}$$

  So $x = 2, -2$.

- Now we use implicit differentiation to get y' in terms of x, y:

$$2x + (y+1)e^y \frac{\mathrm{d}y}{\mathrm{d}x} + e^y \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

- Now set $y = 0$ to get

$$\begin{aligned} 2x + (0+1)e^0 \frac{\mathrm{d}y}{\mathrm{d}x} + e^0 \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ 2x + \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ x + &= -\frac{\mathrm{d}y}{\mathrm{d}x} \end{aligned}$$

- So at $(x, y) = (2, 0)$ we have $y' = -2$,
- and at $(x, y) = (-2, 0)$ we have $y' = 2$.

Marking scheme:

- 1 mark for finding BOTH $x = 2, -2$ as the x-values when $y = 0$ on the given curve.
- 2 marks for implicit differentiation (they do not need to isolate y').
- Note that students can get one of the above without the other. ie if students implicit-diff but not work out x, then they get 1 mark. Similarly if students get both x but do not implicit-diff, they get 1 mark.
- 1 mark for computing at least one of $y' = -2$ at $x = 2$, or $y' = 2$ at $x = -2$.

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your **simplified** answers in the boxes.
 Marking scheme: 1 for each correct, 0 otherwise

 (a) Differentiate $\sin\left(e^{\sin x}\right)$.

 Answer: $\cos(e^{\sin x}) \cdot e^{\sin x} \cdot \cos x$

 Solution: This requires us to apply the chain rule twice.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin\left(e^{\sin x}\right)\right] &= \cos\left(e^{\sin x}\right) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\sin x}\right] \\ &= \cos\left(e^{\sin x}\right) \cdot e^{\sin x}\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] \\ &= \cos\left(e^{\sin x}\right) \cdot e^{\sin x} \cdot \cos x\end{aligned}$$

 (b) If $\cos(x^2 + y^2) = x + y$ compute $\frac{\mathrm{d}y}{\mathrm{d}x}$.

 Answer: $y' = -\dfrac{1 + 2x\sin(x^2 + y^2)}{1 + 2y\sin(x^2 + y^2)}$

 Solution: Differentiate the equation

$$\begin{aligned}-\sin(x^2 + y^2) \cdot (2x + 2yy') &= 1 + y' \\ y'(1 + 2y\sin(x^2 + y^2)) &= -(1 + 2x\sin(x^2 + y^2)) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{1 + 2x\sin(x^2 + y^2)}{1 + 2y\sin(x^2 + y^2)}\end{aligned}$$

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

 (a) Differentiate $f(x) = (\sin x)^{\tan x}$.

 Solution: We use logarithmic differentiation; so

$$\log f(x) = \tan(x)\log(\sin x)$$

 Then differentiate to obtain

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \tan x \cdot \frac{\cos x}{\sin x} + \sec^2(x)\log\sin x \\ &= 1 + \frac{\log\sin x}{\cos^2 x}\end{aligned}$$

In conclusion:

$$f'(x) = f(x) \cdot \left(1 + \frac{\log \sin x}{\cos^2 x}\right) \text{ or equivalently}$$
$$= (\sin x)^{\tan x} \cdot \left(1 + \frac{\log \sin x}{\cos^2 x}\right)$$

Marking scheme:

- 1 mark for using logarithmic differentiation, i.e. write that $\log(f(x)) = \tan x \log \sin x$ — provided they also write that the derivative of $\log f(x)$ is $f'(x)/f(x)$. It is not enough just to write "Use log-diff (or similar)" they have to get to $f'(x)/f(x) = blah$ or $f'(x) = f(x) \times blah$.
- 1 mark for correct derivative for $f(x)$; accept as correct answer also $f'(x) = f(x) \cdot blah$.

(b) Consider a function of the form $f(x) = Ae^{kx}$ where A and k are constants. If $f(0) = -2$ and $f(7) = -3$, find the constants A and k.

Solution: The two pieces of information give us

$$f(0) = A = -2 \qquad\qquad f(7) = Ae^{7k} = -3$$

Thus we know that $A = -2$ and so $f(7) = -3 = -2e^{7k}$. Hence

$$e^{7k} = \frac{3}{2}$$
$$7k = \log(3/2)$$
$$k = \frac{1}{7} \cdot \log(3/2).$$

Marking scheme:

- One mark for each correct constant.
- This one is supposed to be straight forward.

Long answer question — you must show your work

3. 4 marks If $x^2 + y^2 = x\cos(y) + y\cos(x)$, then find y' at the points where $y = 0$. You must justify your answer.

Solution:

- First we find the x-coordinates where $y = 0$.

$$\begin{aligned} x^2 + 0 &= x\cos 0 + 0 = x \\ x^2 - x &= 0 \\ x(x-1) &= 0 \end{aligned}$$

So $x = 0, 1$.

- Now we use implicit differentiation to get y' in terms of x, y:

$$\begin{aligned} x^2 + y^2 &= x\cos(y) + y\cos(x) \\ 2x + 2yy' &= xy'(-\sin y) + \cos y + y(-\sin x) + y'\cos x \end{aligned}$$

- Now set $y = 0$ to get

$$\begin{aligned} 2x &= -xy'\sin 0 + \cos 0 + y'\cos x \\ 2x &= 1 + y'\cos x \\ y' &= \frac{2x-1}{\cos x} \end{aligned}$$

- So at $(x, y) = (0, 0)$ we have $y' = -1$.
- and at $(x, y) = (1, 0)$ we have $y' = \sec(1)$.

Marking scheme:

- 1 mark for finding BOTH $x = 0, 1$ as the x-values when $y = 0$ on the given curve.
- 2 marks for implicit differentiation (they do not need to isolate y').
- Note that students can get one of the above without the other. ie if students implicit-diff but not work out x, then they get 1 mark. Similarly if students get both x but do not implicit-diff, they get 1 mark.
- 1 mark for computing at least one of $y' = -1$ at $x = 0$, or $y' = \sec(1)$ at $x = 1$.