First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

   (a) Find the domain of continuity for the function $f(x) = \dfrac{1}{\sqrt{x^2 - 4}}$.

   Answer: $(-\infty, -2) \cup (2, +\infty)$

   Solution: The function is continuous when $x^2 - 4 > 0$, i.e. $(x-2)(x+2) > 0$, which yields the intervals $(-\infty, -2) \cup (2, +\infty)$.

   (b) Evaluate $\displaystyle\lim_{x \to +\infty} \frac{2x^3 + 4x - 3}{7x^2 + 1}$.

   Answer: Diverges to $+\infty$ or DNE

   Solution: We have, after dividing both numerator and denominator by x^3 (which is the highest power of the denominator) that

   $$\frac{2x^3 + 4x - 3}{7x^2 + 1} = \frac{x^3}{x^2} \frac{2 + 4/x^2 - 3/x^3}{7 + 1/x^2} = x \cdot \frac{2 + 4/x^2 - 3/x^3}{7 + 1/x^2}$$

   Hence as $x \to +\infty$ the fraction goes to a positive constant, but $x \to +\infty$.

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

   (a) Evaluate $\displaystyle\lim_{x \to -\infty} \frac{\sqrt{x^2 + 3} - x}{2x - 7}$

   Solution: Pull out the dominant terms:

   $$\frac{\sqrt{x^2 + 3} - x}{2x - 7} = \frac{\sqrt{x^2(1 + 3/x^2)} - x}{x(2 - 7/x)} \qquad \text{since } x < 0,\ \sqrt{x^2} = -x$$

   $$= \frac{-x\sqrt{1 + 3/x^2} - x}{x(2 - 7/x)} = \frac{-\sqrt{1 + 3/x^2} - 1}{(2 - 7/x)}$$

   Now as $x \to -\infty$, $1/x \to 0$ and $1/x^2 \to 0$, so

   $$\lim_{x \to -\infty} \frac{\sqrt{x^2 + 3} - x}{2x - 7} = -\frac{2}{2} = -1.$$

   Marking scheme:

   - 1 mark for realizing that $\sqrt{x^2} = -x$ when $x < 0$.
   - 1 mark for correct answer.

(b) Find a value of c such that the following function is continuous at $x = c$:

$$f(x) = \begin{cases} \sin(cx) & \text{if } x \le c \\ \cos(cx) & \text{if } x > c \end{cases}$$

Solution: The function is continuous for $x \neq c$ since each of those two branches are polynomials. So, the only question is whether the function is continuous at $x = c$; for this we need

$$\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c+} f(x).$$

We compute

$$\begin{aligned} \lim_{x \to c^-} f(x) &= \lim_{x \to c^-} \sin cx = \sin(c^2) \\ f(c) &= \sin(c^2) \\ \lim_{x \to c^+} f(x) &= \lim_{x \to c^+} \cos cx = \cos(c^2) \end{aligned}$$

So we need $\tan(c^2) = 1$. This happens when $c^2 = \pi/4$. So $c = \sqrt{\pi}/2$.

Marking scheme:

- 1 mark for writing the condition for continuity $\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c+} f(x)$.
- 1 mark for solving correctly and finding a solution.
- Students get at most 1 if they do not check $f(c)$ against the limits.

Long answer question — you must show your work

3. 4 marks Show that there exists a real number such that $c^{-3} = \cos(c)$.

Solution: Let $f(x) = x^{-3} - \cos x$. Then $f(x)$ is continuous everywhere except at $x = 0$. In particular, it is continuous for $x \geq 1/2$.

Now f takes positive values on $[1/2, \infty)$:

$$f(1/2) = 8 - \cos(1/2) \geq 8 - 1 = 7 > 0$$

And f takes negative values on $[1/2, \infty)$:

$$f(2\pi) = \frac{1}{8\pi^3} - \cos(2\pi) = \frac{1}{8\pi^3} - 1 < 0$$

since $8\pi^3 > 1$.

So, because $f(x)$ is continuous on $[1/2, \infty)$ and $f(1/2) > 0$ while $f(2\pi) < 0$, then the Intermediate Value Theorem guarantees the existence of a real number $c \in (1/2, 2\pi)$ such that $f(c) = 0$.

Marking scheme:

- 1 mark for constructing a function $f(x)$ as a difference of the two given functions **and** for writing that $f(x)$ is a continuous function on a **correct** interval I.
- 1 mark for finding a value $a \in I$ such that $f(a) < 0$.
- 1 mark for finding a value $b \in I$ such that $f(b) > 0$.
- 1 mark for the correct conclusion which **should** mention that the solution c is in between a and b **and** its existence is justified by the Intermediate Value Theorem.
- If a and b are on different sides of 0, give at most 2 points total.
- Notice that one can also use numbers like 1 since $f(1) = 1 - cos(1) < 1$ — some students will do this without really explaining why $cos(1) < 1$. I This is okay.

First Name: ______________ Last Name: ______________

Student-No: ______________ Section: ______________

Very short answer questions

1. [2 marks] Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

   (a) Find the domain of continuity for the function $f(x) = \sqrt{x^2 - 1}$.

   Answer: $(-\infty, -1] \cup [1, +\infty)$

   Solution: The function is continuous when $x^2 - 1 \geq 0$, i.e. $(x-1)(x+1) \geq 0$, which yields the intervals $(-\infty, -1] \cup [1, +\infty)$.

   (b) Evaluate $\lim_{x \to +\infty} \frac{5x^2 - 3x + 1}{3x^2 + x + 7}$.

   Answer: $\frac{5}{3}$

   Solution: We have, after dividing both numerator and denominator by x^2 (which is the highest power of the denominator) that

   $$\frac{5x^2 - 3x + 1}{3x^2 + x + 7} = \frac{5 - \frac{3}{x} + \frac{1}{x^2}}{3 + \frac{1}{x} + \frac{7}{x^2}}.$$

   Since $1/x \to 0$ and also $1/x^2 \to 0$ as $x \to +\infty$, we conclude that

   $$\lim_{x \to +\infty} \frac{5x^2 - 3x + 1}{3x^2 + x + 7} = \frac{5}{3}.$$

Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.

   (a) Evaluate $\lim_{x \to -\infty} \frac{3x + 5}{\sqrt{x^2 + 5} - x}$

   Solution: We divide by the highest power of the denominator, which is x and note that

   $$\frac{\sqrt{x^2 + 5}}{x} = -\sqrt{\frac{x^2 + 5}{x^2}} = -\sqrt{1 + \frac{5}{x^2}}.$$

   Since $1/x \to 0$ and also $1/x^2 \to 0$ as $x \to -\infty$, we conclude that

   $$\lim_{x \to -\infty} \frac{3x + 5}{\sqrt{x^2 + 5} - x} = \lim_{x \to -\infty} \frac{3 + \frac{5}{x}}{-\sqrt{1 + \frac{5}{x^2}} - 1} = \frac{3}{-1 - 1} = -\frac{3}{2}.$$

   Marking scheme:

- 1 mark for realizing that $\frac{\sqrt{x^2+5}}{x} = -\sqrt{\frac{x^2+5}{x^2}}$.
- 1 mark for correct answer.

(b) Find all values of c such that the following function is continuous at $x = c$:

$$f(x) = \begin{cases} 8 - cx & \text{if} \quad x \leq c \\ x^2 & \text{if} \quad x > c \end{cases}$$

Solution: The function is continuous for $x \neq c$ since each of those two branches are polynomials. So, the only question is whether the function is continuous at $x = c$; for this we need

$$\lim_{x\to c^-} f(x) = f(c) = \lim_{x\to c+} f(x).$$

We compute

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^-} 8 - cx = 8 - c^2;$$

$$f(c) = 8 - c \cdot c = 8 - c^2 \text{ and}$$

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^+} x^2 = c^2.$$

So, we need $8 - c^2 = c^2$, which yields $c^2 = 4$, i.e. $c = -2$ or $c = 2$. **Marking scheme:**

- 1 mark for writing the condition for continuity $\lim_{x\to c^-} f(x) = f(c) = \lim_{x\to c+} f(x)$.
- 1 mark for solving correctly and finding **both** solutions $c = -2$ and $c = 2$.
- Students get at most 1 if they do not check $f(c)$ against the limits.

Long answer question — you must show your work

3. 4 marks Show that there exists at least one real number c such that $2\tan(c) = c + 1$.

Solution: We let $f(x) = 2\tan(x) - x - 1$. Then $f(x)$ is a continuous function on the interval $(-\pi/2, \pi/2)$ since $\tan(x) = \sin(x)/\cos(x)$ is continuous on this interval, while $x + 1$ is a polynomial and therefore continuous for all real numbers.

We find a value $a \in (-\pi/2, \pi/2)$ such that $f(a) < 0$. We observe immediately that $a = 0$ works since

$$f(0) = 2\tan(0) - 0 - 1 = 0 - 1 = -1 < 0.$$

We find a value $b \in (-\pi/2, \pi/2)$ such that $f(b) > 0$. We see that $b = \pi/4$ works since

$$f(\pi/4) = 2\tan(\pi/4) - \pi/4 - 1 = 2 - \pi/4 - 1 = 1 - \pi/4 = (4 - \pi)/4 > 0,$$

because $3 < \pi < 4$.

So, because $f(x)$ is continuous on $[0, \pi/4]$ and $f(0) < 0$ while $f(\pi/4) > 0$, then the Intermediate Value Theorem guarantees the existence of a real number $c \in (0, \pi/4)$ such that $f(c) = 0$.

Marking scheme:

- 1 mark for constructing a function $f(x)$ as a difference of the two given functions **and** for writing that $f(x)$ is a continuous function on a **correct** interval I (most correct choices would be $(-\pi/2, \pi/2)$).
- 1 mark for finding a value $a \in I$ such that $f(a) < 0$.
- 1 mark for finding a value $b \in I$ such that $f(b) > 0$.
- 1 mark for the correct conclusion which **should** mention that the solution c is in between a and b **and** its existence is justified by the Intermediate Value Theorem.
- If a and b are on different sides of a singularity (eg $\pi/2$), give at most 2 points total.

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

   (a) Describe all points for which the function is continuous: $f(x) = \dfrac{1}{x^2-1}$.

   Answer: All reals except 1,-1

   Solution: The function is continuous over its domain, which is all numbers except positive and negative one.

   (b) Evaluate $\displaystyle\lim_{x\to+\infty} \frac{5x^2+10}{3x^3+2x^2+x}$.

   Answer: 0

   Solution: We have, after dividing both numerator and denominator by x^3 (which is the highest power) that

$$\frac{5x^2+10}{3x^3+2x^2+x} = \frac{\frac{5}{x}+\frac{10}{x^3}}{3+\frac{2}{x}+\frac{1}{x^2}}.$$

   Since $1/x \to 0$, $1/x^2 \to 0$, and $1/x^3 \to 0$ as $x \to +\infty$, we conclude that

$$\lim_{x\to+\infty} \frac{5x^2+10}{3x^3+2x^2+x} = 0.$$

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

   (a) Evaluate $\displaystyle\lim_{x\to-\infty} \frac{5x+7}{\sqrt{4x^2+15}-x}$

   Solution: We divide by the highest power of the denominator, which is x and note that

$$\frac{\sqrt{4x^2+15}}{x} = -\sqrt{\frac{4x^2+15}{x^2}} = -\sqrt{4+\frac{15}{x^2}}.$$

   Since $1/x \to 0$ and also $1/x^2 \to 0$ as $x \to -\infty$, we conclude that

$$\lim_{x\to-\infty} \frac{5x+7}{\sqrt{4x^2+15}-x} = \lim_{x\to-\infty} \frac{5+\frac{7}{x}}{-\sqrt{4+\frac{15}{x^2}}-1} = \frac{5}{-2-1} = -\frac{5}{3}.$$

   Marking scheme:

- 1 mark for realizing that $\frac{\sqrt{4x^2+15}}{x} = -\sqrt{\frac{4x^2+15}{x^2}}$.
- 1 mark for correct answer.

(b) Find all values of c such that the following function is continuous at $x = c$:

$$f(x) = \begin{cases} 6 - cx & \text{if} \quad x \le 2c \\ x^2 & \text{if} \quad x > 2c \end{cases}$$

Solution: The function is continuous for $x \neq c$ since each of those two branches are polynomials. So, the only question is whether the function is continuous at $x = c$; for this we need

$$\lim_{x\to 2c^-} f(x) = f(2c) = \lim_{x\to 2c+} f(x).$$

We compute

$$\lim_{x\to 2c^-} f(x) = \lim_{x\to 2c^-} 6 - cx = 6 - 2c^2;$$

$$f(2c) = 6 - c \cdot 2c = 6 - 2c^2 \text{ and}$$

$$\lim_{x\to 2c^+} f(x) = \lim_{x\to 2c^+} x^2 = 4c^2.$$

So, we need $6 - 2c^2 = 4c^2$, which yields $c^2 = 1$, i.e. $c = -1$ or $c = 1$. **Marking scheme:**

- 1 mark for writing the condition for continuity $\lim_{x\to c^-} f(x) = f(c) = \lim_{x\to c+} f(x)$.
- 1 mark for solving correctly and finding **both** solutions $c = -1$ and $c = 1$.
- Students get at most 1 if they do not check $f(c)$ against the limits.

Long answer question — you must show your work

3. 4 marks Show that there exists at least one real number c such that $3^c = c^2$.

Solution: We let $f(x) = 3^x - x^2$. Then $f(x)$ is a continuous function, since both 3^x and x^2 are continuous for all real numbers.

We find a value a such that $f(a) > 0$. We observe immediately that $a = 0$ works since

$$f(0) = 3^0 - 0 = 1 > 0.$$

We find a value b such that $f(b) < 0$. We see that $b = -1$ works since

$$f(-1) = \frac{1}{3} - 1 = -\frac{2}{3} < 0.$$

So, because $f(x)$ is continuous on $(-\infty, \infty)$ and $f(0) > 0$ while $f(-1) < 0$, then the Intermediate Value Theorem guarantees the existence of a real number $c \in (-1, 0)$ such that $f(c) = 0$.

Marking scheme:

- 1 mark for constructing a function $f(x)$ as a difference of the two given functions **and** for writing that $f(x)$ is a continuous function.
- 1 mark for finding a value a such that $f(a) < 0$.
- 1 mark for finding a value b such that $f(b) > 0$.
- 1 mark for the correct conclusion which **should** mention that the solution c is in between a and b **and** its existence is justified by the Intermediate Value Theorem.

First Name: ______________ Last Name: ______________

Student-No: ______________ Section: ______________

Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

   (a) Find the domain of continuity for the function $f(x) = \sqrt{4 - x^2}$.

   Answer: $[-2, 2]$

   Solution: The function is continuous when $4 - x^2 \geq 0$, i.e. $(2-x)(2+x) \geq 0$, which yields the intervals $[-2, 2]$.

   (b) Evaluate

   $$\lim_{x \to +\infty} \frac{\sqrt{2x^4 + 4x - 3}}{3x^2 + 1}.$$

   Answer: $\sqrt{2}/3$

   Solution: We have, after dividing both numerator and denominator by x^2 (which is the highest power of the denominator) that

   $$\frac{\sqrt{2x^4 + 4x - 3}}{3x^2 + 1} = \frac{\sqrt{x^4(2 + 4/x^3 - 3/x^4)}}{x^2(3 + 1/x^2)} = \frac{x^2\sqrt{2 + 4/x^3 - 3/x^4}}{x^2(3 + 1/x^2)} = \frac{\sqrt{2 + 4/x^3 - 3/x^4}}{3 + 1/x^2}$$

   Hence as $x \to +\infty$ the numerator goes to $\sqrt{2}$ and the denominator to 3.

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.

   (a) Evaluate $\displaystyle\lim_{x \to -\infty} \frac{\sqrt{9x^2 + x} - 3x}{x + 1}$

   Solution: Pull out the dominant terms:

   $$\frac{\sqrt{9x^2 + x} - 3x}{x + 1} = \frac{\sqrt{x^2(9 + 1/x^2)} - 3x}{x(1 + 1/x)} = \frac{-x\sqrt{9 + 1/x^2} - 3x}{x(1 + 1/x)}$$

   $$= \frac{-\sqrt{9 + 1/x^2} - 3}{(1 + 1/x)} \quad \text{since } \sqrt{x^2} = -x \text{ when } x < 0$$

   Now as $x \to -\infty$

   $$\lim_{x \to -\infty} \frac{\sqrt{9x^2 + x} - 3x}{x + 1} = \frac{-\sqrt{9} - 3}{1} = -6.$$

   Marking scheme:

   - 1 mark for realizing that $\sqrt{x^2} = -x$ when $x < 0$.
   - 1 mark for correct answer.

(b) Find a value of c such that the following function is continuous at $x = c$:

$$f(x) = \begin{cases} \sin(x)\cos(x) & \text{if} \quad x \le c \\ \cos(x) & \text{if} \quad x > c \end{cases}$$

Solution: The function is continuous for $x \neq c$ since each of those two branches are polynomials. So, the only question is whether the function is continuous at $x = c$; for this we need

$$\lim_{x\to c^-} f(x) = f(c) = \lim_{x\to c+} f(x).$$

We compute

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^-} \sin x \cos x = \sin c \cdot \cos c$$
$$f(c) = \sin c \cdot \cos c$$
$$\lim_{x\to c^+} f(x) = \lim_{x\to c^+} \cos x = \cos(c)$$

So we need $\cos c = \sin c \cos x$. So this happens when $\cos c = 0$ – that is $c = \pi/2$ (there are other solutions).

Marking scheme:

- 1 mark for writing the condition for continuity $\lim_{x\to c^-} f(x) = f(c) = \lim_{x\to c+} f(x)$.
- 1 mark for solving correctly and finding a solution.
- Students get at most 1 if they do not check $f(c)$ against the limits.

Long answer question — you must show your work

3. 4 marks Show that there exists a real number c such that $2^c = 2\tan(\pi c)$.

Solution: Let $f(x) = 2^x - 2\tan(\pi x)$. Then $f(x)$ is continuous on $(-1/2, 1/2)$ (since that is within the domain of the tangent function).

Now f takes positive values on this interval:

$$f(0) = 2^0 - 2\tan 0 = 1 - 0 = 1.$$

And f takes negative values on this interval:

$$f(1/4) = 2^{1/4} - 2\tan(\pi/4) = 2^{1/4} - 2$$

since $2^{1/4} < 2$.

So, because $f(x)$ is continuous on $[0, 1/4]$ and $f(0) < 0$ while $f(1/4) > 0$, then the Intermediate Value Theorem guarantees the existence of a real number $c \in (0, 1/4)$ such that $f(c) = 0$.

Marking scheme:

- 1 mark for constructing a function $f(x)$ as a difference of the two given functions **and** for writing that $f(x)$ is a continuous function on a **correct** interval I.
- 1 mark for finding a value $a \in I$ such that $f(a) < 0$.
- 1 mark for finding a value $b \in I$ such that $f(b) > 0$.
- 1 mark for the correct conclusion which **should** mention that the solution c is in between a and b **and** its existence is justified by the Intermediate Value Theorem.
- If a and b are on different sides of a singularity (eg 1/2), give at most 2 points total.