D-MODULES: EXERCISE SHEET 1

OCTOBER 26, 2020

Exercise 1. Consider the closed embedding $i\colon \mathbb{A}^{n-k} \hookrightarrow \mathbb{A}^n$ as $x_1 = \cdots = x_k = 0$. Show that

$$\mathscr{D}_{\mathbb{A}^{n-k}\to\mathbb{A}^n} \cong \mathscr{D}_{\mathbb{A}^{n-k}} \otimes_{\mathbb{C}} \mathbb{C}[\partial_1, \ldots, \partial_k]$$

as a left $\mathscr{D}_{\mathbb{A}^{n-k}}$-module.

Exercise 2. Left $f\colon X \to Y$ and $g\colon Y \to Z$ be morphism of smooth complex varieties.

(i) Show that there is a canonical isomorphism $(g \circ f)^! \cong f^! \circ g^!$.

(ii) Show that for any $\mathscr{M} \in \mathrm{D}^-_{\mathrm{qc}}(\mathscr{D}_Y^{\mathrm{op}})$ and $\mathscr{N} \in \mathrm{D}^{\mathrm{b}}(f^{-1}\mathscr{D}_X)$ there is a canonical isomorphism

$$\mathscr{M} \overset{\mathbb{L}}{\otimes}_{\mathscr{D}_Y} \mathbb{R}f_*(\mathscr{N}) \xrightarrow{\sim} \mathbb{R}f_*(f^{-1}\mathscr{M} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} \mathscr{N}).$$

(*Hint:* After constructing the morphism, reduce to a local setting and pick a free resolution of $\mathscr{M}$.)

(iii) Show that there is a canonical isomorphism $(g \circ f)_\bullet \cong g_\bullet \circ f_\bullet$.

Recall that for a closed immersion $i\colon Z \hookrightarrow X$ of smooth varieties there exists a locally free *Koszul resolution*

$$0 \to \mathscr{K}_d \to \ldots \to \mathscr{K}_1 \to \mathscr{K}_0 \to \mathscr{O}_Z \to 0$$

of the $i^{-1}\mathscr{O}_X$-module $\mathscr{O}_Z$, where $d = \operatorname{codim}_X Z$. In local coordinates $\{x_i, \partial_i\}$ on X, if $Z = \{x_1 = \cdots = x_d = 0\}$, one has

$$\mathscr{K}_j = \bigwedge^j \Big(\bigoplus_{k=1}^d i^{-1}\mathscr{O}_X dx_k \Big)$$

with differential $d\colon \mathscr{K}_j \to \mathscr{K}_{j-1}$ given by

$$d(f dx_{k_1} \wedge \cdots \wedge dx_{k_j}) = \sum_{p=1}^{j} (-1)^{p+1} y_{k_p} f dx_{k_1} \wedge \cdots \wedge \widehat{dx_{k_p}} \wedge \cdots \wedge dx_{k_j}.$$

The sheaf $\mathscr{K}_d$ is a locally free $i^{-1}\mathscr{O}_X$-bundle of rank one and there exists a canonical perfect pairing $\mathscr{K}_j \otimes_{i^{-1}\mathscr{O}_X} \mathscr{K}_{d-j} \to \mathscr{K}_d$.

Exercise 3. Let $i\colon Z \hookrightarrow X$ be a closed immersion of smooth complex varieties.

(i) Using the Koszul resolution, show that there exists a canonical isomorphism

$$\mathbb{R}\mathscr{H}om_{i^{-1}\mathscr{D}_X^{\mathrm{op}}}(\mathscr{D}_{Z\to X}, i^{-1}\mathscr{D}_X) \cong \mathscr{D}_{X\leftarrow Z}[-d].$$

(ii) Deduce from this that for $\mathscr{M} \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$ there exists a canonical isomorphism

$$i^! \mathscr{M} \cong \mathbb{R}\mathscr{H}\!om_{i^{-1}\mathscr{D}_X}(\mathscr{D}_{X\leftarrow Z}, i^{-1}\mathscr{M}).$$

Exercise 4. Consider the closed embedding $i\colon \mathbb{A}^{n-k} \hookrightarrow \mathbb{A}^n$ as $x_1 = \cdots = x_k = 0$. For $\mathscr{M} \in \mathbf{Mod}(\mathscr{D}_{\mathbb{A}^{n-k}})$ describe $i_\bullet \mathscr{M}$.

Exercise 5. Let $\mathscr{M} \in \mathbf{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ be a coherent D-module endowed with a good filtration $F_\bullet$. For any fixed integer k and $0 \le p \le \dim X$ set

$$\mathrm{Sp}_k^{-p}(\mathscr{M}) = \mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^p \Theta_X \otimes_{\mathscr{O}_X} F_{k-p}\mathscr{M}.$$

(i) Define a differential $d\colon \mathrm{Sp}_k^{-p}\mathscr{M} \to \mathrm{Sp}_k^{-p+1}\mathscr{M}$ turning $(\mathrm{Sp}_k^\bullet(\mathscr{M}), d)$ into a complex.

(ii) Show that for any sufficiently large integer k this complex is a resolution of $\mathscr{M}$.