Lecture 8

Def: X – smooth alg variety

A holonomic D-module $\mathcal{M}$ on X is called <u>regular</u> if every composition factor of $\mathcal{M}$ is of the form $L(Y, \mathcal{N})$ (with Y connected smooth loc. closed subvariety, $Y \hookrightarrow X$ affine) with $\mathcal{N}$ a (simple) regular integrable connection.

$\mathrm{Mod}_{rh}(X)$ = cat. of reg. hol. module

$D^b_{rh}(X) = \{ \mathcal{M} \in D^b_{hol}(X) \; : \; H^i(\mathcal{M}) \in \mathrm{Mod}_{rh}(X) \}.$

Ex: $X = C$ is a curve $\Rightarrow$ Y is a point $\rightsquigarrow$ no restrictions

or Y is dense open subset

$\rightsquigarrow$ $L(Y, \mathcal{V})$ supported on X

$\Rightarrow$ $\mathcal{M}$ on C is regular if there exists a dense open subset $U \subseteq X$ s.t $\mathcal{M}|_U$ is a regular integrable connection.

Theorem (Curve testing criterion): $\mathcal{M} \in \mathrm{Mod}_{hol}(D_X)$. TFAE

i) $\mathcal{M}$ is regular

ii) $\forall\, i_C : C \hookrightarrow X$, with C a smooth curve, i_C loc. closed embedding the restriction $i_C^! \mathcal{M}$ is regular.

iii) $\forall\, f : C \to X$, with C a smooth curve the pullback $f^! \mathcal{M}$ is regular.

Thm: $f : X \to Y$ then $f^!, f_!, f^\bullet, f_\bullet, \mathbb{D}$ preserves regularity.

Thm: $f : X \to Y$ then $f^!, f_!, f^\bullet, f_\bullet, \mathbb{D}$ commute with DR_X on the regular subcategory.

e.g. $DR_X \circ f^\bullet \simeq f^{-1} \circ DR_Y$

Thm (Riemann-Hilbert Correspondence):

The functor

$$DR_X : D^b_{rh}(D_X) \longrightarrow D^b_{constr}(X)$$

is an equivalence.

Sketch of proof: Have to show that DR_X is

1) fully faithful
2) essentially surjective

1) wts $\quad R\mathcal{H}om_{D_X}(\mathcal{M}, \mathcal{N}) \simeq R\mathcal{H}om_{\mathbb{C}_{X^{an}}}(DR_X(\mathcal{M}), DR_X(\mathcal{N}))$

$p : X \to pt \quad , \quad \Delta : X \to X \times X$ diagonal

$$\mathbb{R}\mathrm{Hom}_{D_X}(\mathcal{M},\mathcal{N}) = p_\cdot\left(\mathbb{D}\mathcal{M} \overset{\mathbb{L}}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{N}\right)^{[-\dim X]} = p_\cdot \Delta^!(\mathbb{D}\mathcal{M} \boxtimes \mathcal{N})$$

$$\text{IS}^{\text{by Thm above.}}$$

$$\mathbb{R}\mathrm{Hom}_{\mathbb{C}_{X^{an}}}(\mathrm{DR}(\mathcal{M}), \mathrm{DR}(\mathcal{N})) = \mathbb{R}p_* \, \Delta^!\left(\mathbb{D}^{top}\mathrm{DR}(\mathcal{M}) \boxtimes \mathrm{DR}(\mathcal{N})\right)$$

2) it suffices to show that some generating set is in the image. Take $\mathbb{R}i_{Y,*}(\mathcal{F})$, $i_Y : Y \hookrightarrow X$ loc closed embedding of smooth subvariety and $\mathcal{F}$ a local system on Y.

This an immediate consequence of Deligne's Riemann-Hilbert correspondence. $\boxed{}$

Question: Is $DR_X(\mathrm{Mod}_{rh}(\mathcal{D}_X)) \subseteq \mathrm{Constr}(\mathbb{C}_{X^{an}})$?

Answer: No, e.g $X = \mathbb{A}^1$

$$DR_X(\mathcal{O}_X) = \mathbb{C}_{X^{an}}[1]$$

$$DR_X(\mathcal{D}_X / \mathcal{D}_X x) = \mathbb{C}_0$$

$$\left[\mathcal{D}_X \xrightarrow{x} \mathcal{D}_X\right]$$

Perverse sheaves

First: t-structures

Def: $\mathcal{T}$ - triangulated category

A <u>t-structure</u> on $\mathcal{T}$ is pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of full subcategories of $\mathcal{T}$ s.t.

i) $\mathcal{T}^{\leq 0}[1] \subseteq \mathcal{T}^{\leq 0}$, $\mathcal{T}^{\geq 0}[-1] \subseteq \mathcal{T}^{\geq 0}$

ii) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}^{\leq 0}, \underbrace{\mathcal{T}^{\geq 1}}_{\mathcal{T}^{\geq 0}[-1]}) = 0$

$\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n]$

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iii) $X \in \mathcal{T}$ $\exists X_{\leq 0} \in \mathcal{T}^{\leq 0}$ and $X_{\geq 1} \in \mathcal{T}^{\geq 1}$ and a d.t.

$$X_{\leq 0} \longrightarrow X \longrightarrow X_{\geq 1} \xrightarrow{+1}$$

Ex: $\mathcal{T} = D(A)$, A - abelian

$$\mathcal{T}^{\leq 0} = \{ X \in \mathcal{T} : H^i(X) = 0 \text{ for } i > 0 \}$$
$$\mathcal{T}^{\geq 0} = \{ X \in \mathcal{T} : H^i(X) = 0 \text{ for } i < 0 \}$$

this is the standard t-structure.

Thm: $\mathcal{T}$ - triang. cat with t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$

i) The inclusion $\mathcal{T}^{\leq n} \hookrightarrow \mathcal{T}$ has a right adjoint $\tau^{\leq n} : \mathcal{T} \to \mathcal{T}^{\leq n}$

ii) —— " ——— $\mathcal{T}^{\geq n} \hookrightarrow \mathcal{T}$ has a left adjoint $\tau^{\geq n} : \mathcal{T} \to \mathcal{T}^{\geq n}$

truncation functors

iii) Any $X \in \mathcal{T}$ fits into a d.t

$$\tau^{\leq 0} X \longrightarrow X \longrightarrow \tau^{\geq 1} X \xrightarrow{+1}$$

adjunction morphisms.

iv) $\mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0} = \mathcal{T}^{\heartsuit}$ is an abelian category

$\uparrow$ heart of the t-structure.

[Warning: in general $\mathcal{T} \neq D^*(\mathcal{T}^{\heartsuit})$]

v) $H^0 = \tau^{\leq 0} \circ \tau^{\geq 0} \simeq \tau^{\geq 0} \circ \tau^{\leq 0} : \mathcal{T} \longrightarrow \mathcal{T}^{\heartsuit}$

is a cohomological functor:

$$X \to Y \to Z \xrightarrow{+1}$$

$$\rightsquigarrow \dots \to H^{n-1}(Z) \longrightarrow H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X) \to \dots$$

is long exact, where $H^n(X) = H^0(X[n])$.

Def: Define full subcategories of $D^b_{constr}(X)$ by

$${}^pD^{\leq 0}_{constr}(X) = \{ \mathcal{F} \in D_{constr}(X) : \dim \operatorname{supp} H^j(\mathcal{F}) \leq -j \quad \forall j \in \mathbb{Z} \}$$

$${}^pD^{\geq 0}_{constr}(X) = \{ \mathcal{F} \in D_{constr}(X) : \dim \operatorname{supp} H^j(\mathbb{D}\mathcal{F}) \leq -j \quad \forall j \in \mathbb{Z} \}$$

Thm [BBD] This is a t-structure on $D^b_{constr}(X)$.

It's called the perverse t-structure. Objects in the heart are called perverse sheaves, $\operatorname{Perv}(X) = {}^pD^{\leq 0}_c(X) \cap {}^pD^{\geq 0}_c(X)$.

Lem: $${}^pD^{\leq 0}_{constr}(X) = \{ \mathcal{F} : H^j(i^{-1}_{X_\alpha} \mathcal{F}) = 0 \quad \forall j > -\dim X_\alpha \}$$

for all $i : X_\alpha \hookrightarrow X$ s.t. $\mathcal{F}|_{X_\alpha}$ is loc. const

$${}^pD^{\geq 0}_{const}(X) = \{ \mathcal{F} : H^j(i^!_{X_\alpha} \mathcal{F}) = 0 \quad \forall j < -\dim X_\alpha \}$$

<u>Thm:</u> The functor DR_X matches the standard t-structure on $D^b_{rh}(\mathcal{D}_X)$ with the perverse t-structure on $D^b_{constr}(X)$,

i.e. $DR_X : \mathrm{Mod}_{rh}(X) \xrightarrow{\sim} \mathrm{Perv}(X)$.