DERIVED CATEGORIES OF COHERENT SHEAVES

NOTES

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CONTENTS

1. INTRODUCTION

It has been understood for a long time that the cohomological theory of coherent sheaves is of fundamental importance to algebraic geometry. Indeed, this realization can be seen as the beginning of "modern" algebraic geometry [S1; G2].

Much more recently, the derived category of coherent sheaves itself has become an object of study. Perhaps the first major impetus for this development was Kontsevich's homological mirror symmetry [K1]. Since then the study derived categories of schemes has bloomed in many different directions.

It is now understood that the derived category is an important invariant of any algebraic variety [BO2], spawning new branches of non-commutative geometry, where (tensor) triangulated categories are the main object of study. Derived categories of coherent sheaves – potentially with addition structure – are now of fundamental importance in a variety of fields, including birational geometry [B6] and representation theory [B4]. In parallel, many efforts were made to understand the structure of these categories, such as Kuznetsov's homological projective duality [K4].

These notes are for a graduate course on derived categories of coherent sheaves. They aim to serve as an introduction to the subject with plenty of references to the literature. The main goals are to understand some of the technology underlying the aforementioned developments, as well as to look at some of the results. These notes are work-in-progress and will be updated as the course progresses.

We should warn the reader that these notes were not particularly well proof-read. As such some statements may be missing assumptions and some proofs may be over-simplified. The reader is advised to consult the given references to the literature.

2. Recollections on derived categories

Let **A** be an abelian category. Homological algebra tells us that we should look at the category of complexes **Kom**(**A**), but of course this category isn't quite the right thing to look at. For example we want to identify homotopic morphisms. But before we do that let us introduce some additional structure.

Firstly, we have a shift endofunctor: $A^\bullet[1] = A^{\bullet+1}$ with differential multiplied by (-1).

Second, given a morphism of complexes $f\colon A^\bullet \to B^\bullet$ we can form the cone

$$\mathrm{cone}(f) = A^\bullet[1] \oplus B^\bullet$$

with differential given by $\begin{pmatrix} d_A[1] & 0 \\ f & d_B \end{pmatrix}$:

$$\begin{tikzcd}
\cdots \arrow[r, "-d_A^{n-1}"] \arrow[dr, "f^{n-1}"] & A^n \arrow[r, "-d_A^n"] \arrow[dr, "f^n"] \arrow[d, phantom, "\oplus"] & A^{n+1} \arrow[r, "-d_A^{n+1}"] \arrow[dr, "f^{n+1}"] \arrow[d, phantom, "\oplus"] & A^{n+2} \arrow[r, "-d_A^{n+2}"] \arrow[dr, "f^{n+2}"] \arrow[d, phantom, "\oplus"] & \cdots \\
\cdots \arrow[r, "d_B^{n-2}"] & B^{n-1} \arrow[r, "d_B^{n-1}"] & B^n \arrow[r, "d_B^n"] & B^{n+1} \arrow[r, "d_B^{n+1}"] & \cdots
\end{tikzcd}$$

We obtain a triangle of morphisms

$$\begin{tikzcd}
 & A \arrow[dr, "f"] & \\
\mathrm{cone}(f) \arrow[ur, "+1"] & & B \arrow[ll]
\end{tikzcd}$$

Recall that two morphisms of complexes $f, g\colon A \to B$ are homotopic if there exits a collection of maps $h^\bullet\colon A^\bullet \to B^{\bullet-1}$ such that $f - g = d_B h + h d_A$. In this case we write $f \sim g$.

$$\begin{tikzcd}
\cdots \arrow[r, "d_A^{n-2}"] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d, shift right, "f^{n-1}"'] \arrow[d, shift left, "g^{n-1}"] \arrow[dl, dashed, "h^{n-1}"'] & A^n \arrow[r, "d_A^n"] \arrow[d, shift right, "f^n"'] \arrow[d, shift left, "g^n"] \arrow[dl, dashed, "h^n"'] & A^{n+1} \arrow[r, "d_A^{n+1}"] \arrow[d, shift right, "f^{n+1}"'] \arrow[d, shift left, "g^{n+1}"] \arrow[dl, dashed, "h^{n+1}"'] & \cdots \arrow[dl, dashed, "h^{n+2}"'] \\
\cdots \arrow[r, "d_B^{n-2}"] & B^{n-1} \arrow[r, "d_B^{n-1}"] & B^n \arrow[r, "d_B^n"] & B^{n+1} \arrow[r, "d_B^{n+1}"] & \cdots
\end{tikzcd}$$

We write $\mathbf{K}(\mathbf{A})$ for the homotopy category of complexes, i.e. the category with the same objects as **Kom**(**A**) and with morphisms

$$\mathrm{Hom}_{\mathbf{K}(\mathbf{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathbf{Kom}(\mathbf{A})}(A^\bullet, B^\bullet)/\sim .$$

We call any triangle $X \to Y \to Z \to X[1]$ which is isomorphic to a triangle $X \xrightarrow{f} Y \to \mathrm{cone}(f) \to X[1]$ in $\mathbf{K}(\mathbf{A})$ a *distinguished triangle*. The shift functor and the collection distinguished triangles give $\mathbf{K}(\mathbf{A})$ the structure of a *triangulated category*. Thus it is an additive category and satisfies the following axioms.

(TR$_1$) For any object X, the triangle $X \xrightarrow{\mathrm{Id}} X \to 0$ is distinguished. For any morphism $f\colon X \to Y$ there exists a distinguished triangle $X \xrightarrow{f} Y \to Z$ (Z is called a *mapping cone of f*). Any triangle that is isomorphic to a distinguished triangle is distinguished.

(TR$_2$) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is distinguished, then so are the rotated triangles

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1] \quad \text{and} \quad Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z.$$

(TR$_3$) Given two triangles and maps f and g which make the left-most square in the diagram below commute, there exists a (not necessarily unique) morphism h making everything commute

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, dashed, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

(TR$_4$) The octahedral axiom. Given maps $f\colon X \to Y$ and $g\colon Y \to Z$ the mapping cones of f, g and gf fit into a distinguished triangle $\mathrm{cone}(f) \to \mathrm{cone}(gf) \to \mathrm{cone}(g)$, so that all possible diagrams commute. (These diagrams can be drawn in the shape of an octagon.)

Note that it follows from (TR$_3$) that any two mapping cones of a morphism f are isomorphic, but not necessarily uniquely so. In particular, a morphism f is an isomorphism if and only if $\mathrm{cone}(f) = 0$. We also note that $\mathbf{Kom}(\mathbf{A})$ does not satisfy (TR$_1$), as the cone of the identity morphism is only null-homotopic, but not isomorphic to zero.

A functor between triangulated categories $F\colon \mathbf{S} \to \mathbf{T}$ together with an isomorphism $\phi\colon F \circ [1] \cong [1] \circ F$ is called *exact* (or *triangulated*) if it is additive and sends distinguished triangles to distinguished triangles.

A morphism of complexes is called a *quasi-isomorphism* if it induces an isomorphism on cohomology. We want to identify quasi-isomorphic objects and thus formally invert all quasi-isomorphisms.

Definition 2.1. The derived category $\mathrm{D}(\mathbf{A})$ of $\mathbf{A}$ has the same objects as $\mathbf{K}(\mathbf{A})$ and morphisms $X \to Y$ in $\mathrm{D}(\mathbf{A})$ are roofs

$$\begin{tikzcd} & C \arrow[dl, "\text{quis}"', "a"] \arrow[dr, "b"] & \\ A & & B \end{tikzcd}$$

where a is a homotopy class of a quasi-isomorphism $C \to A$ and b is a homotopy class of a morphism $C \to B$.

We write $\mathrm{D}^+(\mathbf{A})$, $\mathrm{D}^-(\mathbf{A})$ and $\mathrm{D}^b(\mathbf{A})$ for the full subcategories of $\mathrm{D}(\mathbf{A})$ consisting of objects $X \in \mathrm{D}(\mathbf{A})$ such that $H^i(X) = 0$ for all $i \ll 0$, resp. all $i \gg 0$, resp. all i with $|i| \gg 0$.

There exists a natural quotient functor $\mathbf{K}(\mathbf{A}) \to \mathrm{D}(\mathbf{A})$. We declare any triangle which is isomorphic to the image of a distinguished triangle under this morphism (i.e., to a triangle of the form $X \xrightarrow{f} Y \to \operatorname{cone} f$) to be a distinguished triangle in $\mathrm{D}(\mathbf{A})$. This gives $\mathrm{D}(\mathbf{A})$ the structure of a triangulated category. The same is true for the various bounded versions, which are full triangulated subcategories of $\mathrm{D}(\mathbf{A})$.

If $F\colon \mathbf{A} \to \mathbf{B}$ is a left exact functor we define its *right derived functor* $RF\colon \mathrm{D}^+(\mathbf{A}) \to \mathrm{D}^+(\mathbf{B})$ by $RF(X) = F(I)$, where I is a complex consisting of injective objects with a quasi-isomorphism $X \xrightarrow{\sim} I$ and $F(I)$ is to be understood component-wise. Similarly, we define the *left derived functor* $LF\colon \mathrm{D}^-(\mathbf{A}) \to \mathrm{D}^-(\mathbf{B})$ of a right exact functor F as $F(P)$ for a projective resolution $P \xrightarrow{\sim} X$.

In particular, we obtain a bifunctor $R\operatorname{Hom}(-,-)$.

Exercise 2.2. Let $A = k[\epsilon]/(\epsilon^2)$ be the dual numbers. Show that in $\mathrm{D}^b(\mathbf{Mod}(A))$ the complexes $0 \to k \xrightarrow{0} k \to 0$ and $0 \to A \xrightarrow{\epsilon} A \to 0$ are not quasi-isomorphic, despite having isomorphic cohomology modules. [Hint: Compute the endomorphism rings of the second complex. Note that this is easy to do since it is a bounded complex of projective A-modules. Deduce that it is indecomposable.]

The above assumes that the category has enough injectives and projectives respectively. For categories of sheaves this is not always the case. Thus one uses F-injective (resp. F-projective) complexes (see for example [w, Theorem 10.5.9]. For example, we can compute the derived tensor product $\otimes^L$ with a flat (e.g. locally free) resolution.

Warning 2.3. The above only applies to the appropriately bounded derived category. Unbounded derived categories can behave in unexpected ways. For example the complex

$$\cdots \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \cdots$$

is an acyclic complex, i.e. it is quasi-isomorphic to the zero complex in $\mathrm{D}(\mathbf{Mod}(\mathbb{Z}/4))$. But while the complex consists of free modules, it cannot be used to compute the derived tensor product: tensoring the complex with $\mathbb{Z}/2$ gives

$$\cdots \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \cdots,$$

which is not acyclic.

In order to work correctly with the unbounded derived category, one has to modify the notion of injective/projective resolution slightly [s2]. We will not have to do any concrete computations in the unbounded derived category, so we will take these results on faith.

Triangulated categories as the underlying structure of derived categories have some drawbacks. For example, cones are only unique up to non-unique isomorphism. It follows that usually one cannot glue derived categories. This is a major problem when defining derived categories of sheaves on stacks. Further it is very much not obvious what the correct definition of a (co)limit in a triangulated category should be. We will run into this problem later on.

For this reason one often considers ∞-categorical enhancements of the derived category, i.e. a pretriangulated dg-category or stable infinity category whose homotopy category is the given derived category. However, as this adds another layer of technical complexity, we will not do so in this course. We will however leave occasional remarks to note where the ∞-categorical perspective yields a clearer picture.

3. CATEGORIES AND FUNCTORS OF (QUASI)COHERENT SHEAVES

We are interested in the bounded derived category of coherent sheaves. Unfortunately, the category of coherent sheaves does not have enough injectives. This is a general phenomenon in homological algebra: one is interested in the derived category of an abelian category $\mathbf{A}$ which is too "small" for some construction (in this case the Godemont resolution). On the other hand $\mathbf{A}$ often sits inside a larger category $\mathbf{B}$ which provides all the needed flexibility. In this case the right approach is to consider the full subcategory of $\mathrm{D}(\mathbf{B})$ whose objects are those complexes with finitely many non-zero cohomology objects all of which are contained in $\mathbf{A}$.

Thus, instead of the category $\mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X))$ we will work with the category

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \{\mathscr{F} \in \mathrm{D}(\mathbf{QCoh}(X)) : H^i(\mathscr{F}) \in \mathbf{Coh}(X) \text{ for all } i \text{ and } H^j(\mathscr{F}) = 0 \text{ for } |i| \gg 0\}.$$

We note that $\mathbf{QCoh}(X)$ has enough injectives [H1, Theorem II.7.18].

In general for abelian categories $\mathbf{A} \subset \mathbf{B}$, the categories $\mathrm{D}^{\mathrm{b}}(\mathbf{A})$ and $\mathrm{D}^{\mathrm{b}}_{\mathbf{A}}(\mathbf{B})$ might be different. Luckily for us, in our case they are actually the same.

Theorem 3.1. *If X is a noetherian scheme the natural functor*

$$\mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X)) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$$

is an equivalence.

Lemma 3.2 ([H2, Exercise II.5.15])**.** *Let X be a noetherian scheme.*

(i) Any quasi-coherent sheaf on X is the union of its coherent subsheaves.

(ii) Let $U \subseteq X$ be open and $\mathscr{F} \in \mathbf{Coh}(U)$. Then there exists a coherent sheaf $\mathscr{G}$ on X with $\mathscr{G}|_U \cong \mathscr{F}$.

It follows from this that if $\mathscr{F} \twoheadrightarrow \mathscr{G}$ is a surjective morphism of quasi-coherent sheaves with $\mathscr{G}$ coherent, then there exists a coherent subsheaf $\mathscr{F}' \subseteq \mathscr{F}$ such that the restricted map $\mathscr{F}' \to \mathscr{G}$ remains surjective. (This is clearly true when X is affine. The global statement follows from the local one by point (ii) above.)

Proof of Theorem 3.1. Let $\mathscr{F}$ be a bounded complex

$$\cdots 0 \to \mathscr{F}^n \to \cdots \to \mathscr{F}^m \to 0 \cdots$$

of quasi-coherent sheaves with coherent cohomology sheaves. By induction suppose $\mathscr{F}^j$ is coherent for $j > i$. Consider the surjections

$$d^i \colon \mathscr{F}^i \twoheadrightarrow \mathrm{im}(d^i) \subseteq \mathscr{F}^{i+1} \quad \text{and} \quad \ker(d^i) \twoheadrightarrow H^i(\mathscr{F}).$$

We can find coherent subsheaves of $\mathscr{F}^i_1 \subseteq \mathscr{F}^i$ and $\mathscr{F}^i_2 \subseteq \ker(d^i) \subseteq \mathscr{F}^i$ such that the restrictions of the above morphisms are still surjective. Now replace $\mathscr{F}^i$ by its subsheaf generated by $\mathscr{F}^i_1$ and $\mathscr{F}^i_2$, and let $\mathscr{F}^{i-1}$ be the preimage under d^{i-1} of the new $\mathscr{F}^i$. Clearly the inclusions induce a quasi-isomorphism of the new complex with the old one and now $\mathscr{F}^i$ is also coherent.

Thus we showed that the functor $\mathrm{D^b}(\mathbf{Coh}(X)) \to \mathrm{D^b_{coh}}(X)$ is essentially surjective. Using general structure theory of derived categories (e.g., [KS, Proposition 1.6.5]), it is not too hard to show that it is also fully faithful. □

Remark 3.3. Instead of $\mathrm{D}(\mathbf{QCoh}(X))$ one might take the full subcategory $\mathrm{D_{qc}}(X)$ of $\mathrm{D}(\mathbf{Mod}(\mathscr{O}_X))$ given by

$$\mathrm{D_{qc}}(X) = \{\mathscr{F} \in \mathrm{D}(\mathbf{Mod}(\mathscr{O}_X)) : H^i(\mathscr{F}) \in \mathbf{QCoh}(X)\}.$$

It is known that at least when X is quasi-compact and separated then these two categories coincide [BN, Corollary 5.5], but usually one should really take the latter category. (See also [H1, Corollary II.7.19] for a proof that the D^+-versions of these categories coincide on any locally noetherian scheme.)

We will come across constructions that only work in categories where all (small) coproducts exist. The category $\mathrm{D^b_{coh}}(X)$ is certainly not of this form – it does not contain infinite direct sums. Thus we will have to consider the category $\mathrm{D_{qc}}(X)$ which is closed under forming coproducts. We do not really like working with this category: its objects are infinite dimensional and it is an unbounded derived category. But it does provide an extra level of flexibility. In order to not derail this course into a course on homological algebra, we will have to take various facts about $\mathrm{D_{qc}}(X)$ on faith. A reference is [S2].

3.1. STANDARD FUNCTORS

Consider a morphism $f\colon X \to Y$. If X is separated and noetherian, then f_* sends quasi-coherent sheaves to quasi-coherent sheaves. We can compute the derived functor Rf_* on $\mathrm{D^-_{qc}}(X)$ by using injective or flasque resolutions. (Of course, in practice this is usually quite hard to do, so one might resort to different methods, like Čech cohomology.) Thus we obtain a derived functor $Rf_*\colon \mathrm{D^+_{qc}}(X) \to \mathrm{D^+_{qc}}(Y)$.

If Y is quasi-compact, then there exists an integer r, depending only on f such that $R^if_*\mathscr{F} = 0$ for all quasi-coherent $\mathscr{F}$ and all $i > r$ [EGA3I, Proposition 1.4.10] and hence Rf_* restricts to the bounded derived categories. Unfortunately, in general f_* does not preserve coherence (e.g., take global sections of $\mathscr{O}_{\mathbb{A}^1}$). However, if f is a proper morphism of separated noetherian schemes, then f_* preserves coherence, as do the higher direct images [EGA3I, Corollaire 1.4.12]. Thus in this case we obtain a functor $Rf_*\colon \mathrm{D^b_{coh}}(X) \to \mathrm{D^b_{coh}}(Y)$.

In the other direction we have the inverse image $f^* = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}$. As f^{-1} is an exact functor, we only need to take the left derived functor of the tensor product. Thus we obtain a functor $Lf^*\colon \mathrm{D^-_{qc}}(X) \to \mathrm{D^-_{qc}}(X)$. We would usually compute this functor using a locally free (and hence in particular flat) resolution. Note that if f is a flat morphism, then essentially by definition f^* is exact. If Y is Noetherian and smooth, then $\mathscr{O}_Y$ has finite weak global dimension and hence f^* induces a functor $Lf^*\colon \mathrm{D^b_{coh}}(Y) \to \mathrm{D^b_{coh}}(Y)$.

The functor f^* is left adjoint to f_* whenever both are defined.

Taking an injective resolution of the second argument, we define $R\,\mathrm{Hom}$ and $R\mathscr{H}om$ on $\mathrm{D}^-_{\mathrm{qc}}(X) \otimes \mathrm{D}^+_{\mathrm{qc}}(X)$. Note that if X is affine, we can compute $R\,\mathrm{Hom}$ with a free resolution of the first argument. Hence we can always compute $R\mathscr{H}om$ with a locally free resolution of the first argument – if such a resolution exists. Similarly, we define $\otimes^L$ on $\mathrm{D}^-_{\mathrm{qc}}(X) \otimes \mathrm{D}^-_{\mathrm{qc}}(X)$ using a flat resolution. As usual, $\otimes^L$ is left adjoint to $R\mathscr{H}om$, i.e.

$$\mathrm{Hom}(\mathscr{F} \otimes^L \mathscr{G}, \mathscr{H}) \cong \mathrm{Hom}(\mathscr{F}, R\mathscr{H}om(\mathscr{G}, \mathscr{H})).$$

If X is smooth and noetherian, then these functors restrict to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. Following [S2], all of these functors can also be defined on the unbounded categories $\mathrm{D}_{\mathrm{qc}}(X)$. The only difficulty is that the internal $\mathscr{H}om$ in $\mathrm{D}_{\mathrm{qc}}(X)$ and $\mathrm{D}(\mathbf{Mod}(\mathscr{O}_X))$ may not agree on unbounded complexes (the existence of the former follows from Theorem 4.3 below).

Notation. From now on – unless otherwise specified – all functors will be derived and we drop the signifiers R and L. The only exception of this rule is that Hom will generally denote the usual Hom in any category.

For future use let us state the *flat base change* isomorphism. It follows (under slightly stronger assumptions) directly from [H2, Theorem III.9.3]. Alternatively, it follows from the projection formula, which we will prove later on (see [STACKS, Tag 08IB]).

Fact 3.4. *Consider a cartesian diagram*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "v"] \arrow[d, "g"] & Y \arrow[d, "f"] \\ X \arrow[r, "u"] & Z \end{tikzcd}$$

with $u\colon X \to Z$ flat and $f\colon Y \to Z$ quasi-compact and quasi-separated. Then we have a canonical isomorphism of functors

$$u^*f_* \xrightarrow{\sim} g_*v^*.$$

Remark 3.5. Switching to derived algebraic geometry would allow us to remove the assumptions of f and g. The price we would have to pay is that we have to take the fiber product $X \times_Z Y$ in a derived sense.

In any case, we always have a morphism $u^*f_* \to g_*v^*$:

$$u^*f_* \to g_*g^*u^*f_* = g_*v^*f^*f_* \to g_*v^*,$$

where we use the (g^*, g_*) and (f^*, f_*) adjunctions to obtain the two arrows.

3.2. PERFECT COMPLEXES

Computing these functors on arbitrary objects can often be quite hard. Suppose however that we understand the functors on some class of "suitably finite" objects. We could then try build bigger objects by taking colimits (e.g. unions and direct sums) of these objects.

Taking (co)limits in derived categories is a rather tricky business, as everything should be taken up to homotopy. That is, we should consider the notion of a homotopy colimit. As we'd prefer not to go too deep down that particular rabbit hole we will limit ourselves to coproducts. These are easy to construct even in the derived category and fortunately suffice for our purposes.

Remark 3.6. The modern way to proceed here is to consider a stable $(\infty, 1)$-categorical enhancement of the derived categories. The usual ∞-categorical notion of (co)limits would then induce the correct homotopy (co)limits on the triangulated categories.

Fact 3.7 ([BN, Section 1])**.** *Suppose $\mathscr{F}_\alpha$ are objects of $\mathrm{D}_{\mathrm{qc}}(X)$. Then the coproduct $\coprod_\alpha \mathscr{F}_\alpha$ is formed degree-wise, i.e. it is given by the complex*

$$\cdots \to \coprod_\alpha \mathscr{F}_\alpha^{n-1} \xrightarrow{\coprod d} \coprod_\alpha \mathscr{F}_\alpha^{n} \xrightarrow{\coprod d} \coprod_\alpha \mathscr{F}_\alpha^{n+1} \to \cdots .$$

Coproducts of distinguished triangles are distinguished triangles.

Definition 3.8. A functor $F\colon \mathbf{C} \to \mathbf{D}$ *respects coproducts* if for any $X = \coprod X_\alpha$ in $\mathbf{C}$ the coproduct $\coprod F(X_\alpha)$ exists and the canonical morphism $\coprod F(X_\alpha) \to F(\coprod X_\alpha)$ is an isomorphism.

Lemma 3.9. *If a functor $F\colon \mathbf{C} \to \mathbf{D}$ has a right adjoint, then F respects coproducts.*

Proof. Let $G\colon \mathbf{D} \to \mathbf{C}$ be the right adjoint. Then for any $Y \in \mathbf{D}$ we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}}(F(\coprod X_\alpha), Y) &= \mathrm{Hom}_{\mathbf{C}}(\coprod X_\alpha, G(Y)) \\
&= \prod \mathrm{Hom}_{\mathbf{C}}(X_\alpha, G(Y)) \\
&= \prod \mathrm{Hom}_{\mathbf{D}}(F(X_\alpha), Y) \\
&= \mathrm{Hom}_{\mathbf{D}}(\coprod F(X_\alpha), Y).
\end{aligned}$$

Thus the result follows from the Yoneda Lemma. □

Corollary 3.10. *The functors f^* and $\otimes$ respect coproducts.* □

Lemma 3.11 ([N1, Lemma 1.4])**.** *Let $f\colon X \to Y$ be a separated morphism of schemes with X quasi-compact and separated. Then the functor $f_*\colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y)$ respects coproducts.*

Sketch of proof. We must show that the canonical morphism

$$\coprod f_*(\mathscr{F}_\alpha) \to f_*(\coprod \mathscr{F}_\alpha)$$

is an isomorphism for any $\mathscr{F}_\alpha \in \mathrm{D}_{\mathrm{qc}}(X)$. The question is local on Y, so we can assume that Y is affine.

If X is also affine, then the statement is trivial. Otherwise we induct on the number of open subsets in an affine cover of X using the distinguished triangle

$$\mathrm{Id} \to j_{U,*}j_U^* \oplus j_{V,*}j_V^* \to j_{U\cap V,*}j_{U\cap V}^*$$

for U and V open subsets. □

In particular it follows that the derived global sections functor respects coproducts and in particular so does

$$H^i(X, -) \colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathbf{Ab}.$$

Our goal is to deduce information from "suitably finite" objects. But what should "suitably finite" mean?

If $X = \operatorname{Spec}(A)$ is affine, the nicest objects in $\mathbf{QCoh}(X)$ are clearly the finitely generated free modules A^n. More generally, for arbitrary X the finite dimensional vector bundles (i.e. locally free $\mathcal{O}_X$-modules) seem to be relatively easy to understand. We give complexes of these a fitting name.

Definition 3.12. A complex $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(X)$ is called *perfect* if, locally on X, it is isomorphic to a bounded complex of finitely generated projective (or equivalently, free) $\mathcal{O}_X$-modules. We write $\mathbf{Perf}(X)$ for the full subcategory consisting of perfect complexes.

We note that the cone of a morphism of perfect complexes is again perfect, so that $\mathbf{Perf}(X)$ is a triangulated subcategory. Clearly we have $\mathbf{Perf}(X) \subseteq \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$.

Example 3.13. The category $\mathbf{Perf}(X)$ is bigger than it would seem at first. For example, if $X = \mathbb{A}^1 = \operatorname{Spec} k[x]$, then the skyscraper k_0 at the origin is quasi-isomorphic to the complex $k[x] \xrightarrow{x} k[x]$ and hence is perfect even though it is not a vector bundle itself. Indeed we have the following statement. ○

Theorem 3.14. *A noetherian scheme is regular if and only if* $\mathbf{Perf}(X) = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$.

Idea of proof. Apply the Auslander–Buchsbaum–Serre homological regularity characterization. □

From an abstract point-of-view, why should we be interested in perfect complexes? To answer this question let us introduce two categorical notions of finiteness.

Definition 3.15. An object X of a triangulated category $\mathbf{T}$ is called *compact* if for any coproduct $\coprod Y_\alpha$ of objects in $\mathbf{T}$ one has

$$\operatorname{Hom}_{\mathbf{T}}(X, \coprod_\alpha X_\alpha) = \coprod_\alpha \operatorname{Hom}_{\mathbf{T}}(X, Y_\alpha).$$

We note that for arbitrary objects X this would only hold for finite coproducts.

Remark 3.16. A priori this notion differs from the usual categorical definition of compactness, where one requires that $\operatorname{Hom}(X, -)$ commutes with all co*limits*. However, it turns out that if $\mathbf{T}$ is the homotopy category of a stable infinity category $\mathbf{S}$, then the above definition already implies that $\operatorname{Hom}_{\mathbf{S}}(X, -)$ commutes with all colimits (taken in the ∞-categorical sense). This is one of the reasons why it often feels more natural to work with an ∞-categorical enhancement of a derived category, rather than with the derived category itself.

Since $\mathcal{H}om(\mathcal{F}, -)$ can be computed using a locally free resolution of $\mathcal{F}$, it follows that if X is quasi-compact and separated, then every perfect complex is a compact object of $\mathrm{D}_{\mathrm{qc}}(X)$ [N1, Example 1.13]. Categorically speaking, the fact that a category has "enough" compact objects is expressed in the follow definition.

Definition 3.17. A triangulated category $\mathbf{T}$ is *compactly generated* if it contains all coproducts and there exists a set C of compacts objects of $\mathbf{T}$ such that any object $X \in \mathbf{T}$ is zero if and only if $\mathrm{Hom}(c, X) = 0$ for all $c \in C$.

Why do we call this "generation"? The reason is the following very useful result.

Fact 3.18 ([N1, Lemma 3.2])**.** *Let* $\mathbf{T}$ *be compactly generated by a set* C *of compact objects. Let* $\mathbf{S}$ *be a full triangulated subcategory of* $\mathbf{T}$ *which contains* C *and is closed under formation of coproducts (taken in* $\mathbf{T}$*). Then* $\mathbf{S} = \mathbf{T}$.

Theorem 3.19 ([N1, Section 2], see also [BFN, Section 3])**.** *Let* X *be a quasi-compact and separated scheme. Then* $\mathrm{D}_{\mathrm{qc}}(X)$ *is compactly generated by* $\mathbf{Perf}(X)$ *and a complex* $\mathscr{F} \in \mathrm{D}_{\mathrm{qc}}(X)$ *is compact if and only if it is perfect.*

Of course the category $\mathrm{D}_{\mathrm{qc}}(X)$ has additional structure: it is symmetric monoidal, i.e. it has a tensor product.

Definition 3.20. An object A of a symmetric monoidal category with unit $\mathbb{1}$ is called *(strongly) dualizable* if there exists an object A^* and unit and counit maps

$$\mathbb{1} \to A^* \otimes A \to \mathbb{1}$$

such that the compositions

$$A \to A \otimes (A^* \otimes A) \cong (A \otimes A^*) \otimes A \to A$$

and

$$A^* \to (A^* \otimes A) \otimes A^* \cong A^* \otimes (A \otimes A^*) \to A^*$$

are the identity on A and A^* respectively.

These formulas are quite akin to those of an adjunction. Indeed A and A^* can bee seen as adjoints in an appropriate category. In particular, A^* is unique up to canonical isomorphism if it exists.

Exercise 3.21. What does this mean in the category of vector spaces?

If the category is in addition closed monoidal, i.e. it has an internal $\mathscr{H}\!om$ right adjoint to $\otimes$, then one sets $A^\vee = \mathscr{H}\!om(A, \mathbb{1})$ and A is dualizable if an only if the canonical map

$$A^\vee \otimes A \to \mathscr{H}\!om(A, A)$$

is an isomorphism. It follows that if A is dualizable, then for any other object B one has an isomorphism $A^\vee \otimes B \cong \mathscr{H}\!om(A, B)$ and that $(A^\vee)^\vee = A$.

Corollary 3.22. *In* $\mathrm{D}_{\mathrm{qc}}(X)$ *perfect complexes, compact objects and dualizable objects coincide.*

Proof. We already know that the compact objects are exactly the perfect complexes. One easily checks locally that every perfect complex is dualizable in $\mathrm{D}_{\mathrm{qc}}(X)$ [STACKS, Tag 08DQ]. Finally, if $\mathscr{F}$ is dualizable, then $\mathscr{H}\!om(\mathscr{F}, -) = \mathscr{F}^\vee \otimes -$ preserves coproducts. As the global sections functor also preserves coproducts it follows that $\mathscr{F}$ is compact. $\square$

As we remarked before, if X is not smooth then $\mathbf{Perf}(X)$ does not coincide with $D^b_{coh}(X)$. In fact Orlov introduced the *category of singularities* $D_{sg}(X) = D^b_{coh}(X)\big/\mathbf{Perf}(X)$ as a measure of how singular X is [O3]. In a similar vein, at least when X is lci, one can associate to each $\mathscr{F} \in D^b_{coh}(X)$ its "singular support", measuring how far $\mathscr{F}$ is from being perfect [AG].

While perfect complex are obviously in many ways quite nice, it can sometimes be bothersome that not all coherent complexes are compact in $D_{qc}(X)$. To rectify this situation, one can enlarge $D_{qc}(X)$ to the category of *ind-coherent sheaves* [G1]. This is a category which is (essentially by definition) compactly generated by $D^b_{coh}(X)$. We will not further discuss these categories here.

3.3. THE PROJECTION FORMULA

So far we have discussed a lot of "categorical nonsense". We will now see how to put it to use to prove a concrete statement.

Theorem 3.23 ([N1, Proposition 5.3])**.** *Let $f\colon X \to Y$ be a morphism of quasi-compact and separated schemes. Then there exists a canonical isomorphism*

$$\mathscr{F} \otimes f_*\mathscr{G} \xrightarrow{\sim} f_*(f^*\mathscr{F} \otimes \mathscr{G})$$

for any $\mathscr{F} \in D_{qc}(Y)$ and $\mathscr{G} \in D_{qc}(X)$.

Proof. By (f^*, f_*)-adjunction we need to produce a morphism

$$f^*(\mathscr{F} \otimes f_*\mathscr{G}) \to f^*\mathscr{F} \otimes \mathscr{G}.$$

Now,

$$f^*(\mathscr{F} \otimes f_*\mathscr{G}) \cong f^*\mathscr{F} \otimes f^*f_*\mathscr{G} \to f^*\mathscr{F} \otimes \mathscr{G},$$

where the first isomorphism is just the fact that f^* is monoidal and the second map is given by (f^*, f_*)-adjunction. We need to show that this morphism is an isomorphism.

We first assume that $\mathscr{F}$ is perfect. Then for any $\mathscr{H} \in D_{qc}(Y)$ we have

$$\begin{aligned}
\mathrm{Hom}(\mathscr{H}, \mathscr{F} \otimes f_*\mathscr{G}) &\cong \mathrm{Hom}(\mathscr{H}, \mathscr{H}om(\mathscr{F}^\vee, f_*\mathscr{G})) && (\mathscr{F} \text{ is dualizable})\\
&\cong \mathrm{Hom}(\mathscr{H} \otimes \mathscr{F}^\vee, f_*\mathscr{G}) && (\text{tensor-hom adjunction})\\
&\cong \mathrm{Hom}(f^*(\mathscr{H} \otimes \mathscr{F}^\vee), \mathscr{G}) && ((f^*, f_*)\text{-adjunction})\\
&\cong \mathrm{Hom}(f^*\mathscr{H} \otimes f^*(\mathscr{F})^\vee, \mathscr{G}) && (f^* \text{ is monoidal})\\
&\cong \mathrm{Hom}(f^*\mathscr{H} \otimes (f^*\mathscr{F})^\vee, \mathscr{G}) && (f^* \text{ is monoidal})\\
&\cong \mathrm{Hom}(f^*\mathscr{H}, \mathscr{H}om((f^*\mathscr{F})^\vee, \mathscr{G})) && (\text{tensor-hom adjunction})\\
&\cong \mathrm{Hom}(f^*\mathscr{H}, f^*\mathscr{F} \otimes \mathscr{G}) && (\text{dualizability})\\
&\cong \mathrm{Hom}(\mathscr{H}, f_*(f^*\mathscr{F} \otimes \mathscr{G})) && ((f^*, f_*)\text{-adjunction}).
\end{aligned}$$

An application of the Yoneda Lemma yields the result.

Let $\mathbf{S}$ be the full subcategory of $D_{qc}(Y)$ consisting of all complexes $\mathscr{F}$ for which the morphism $\mathscr{F} \otimes f_*\mathscr{G} \to f_*(f^*\mathscr{F} \otimes \mathscr{G})$ is an isomorphism for all $\mathscr{G} \in D_{qc}(X)$. We just showed that $\mathbf{Perf}(Y) \subseteq \mathbf{S}$. We already know that all functors in the statement respect shifts, triangles and coproducts. Thus $\mathbf{S} = \mathrm{D}_{\mathrm{qc}}(Y)$ by Fact 3.18. □

An interesting case of the projection formula arises when we take $\mathscr{G} = \mathscr{O}_X$: $f_* f^* \mathscr{F} = \mathscr{F} \otimes f_* \mathscr{O}_X$.

3.4. PROVING IDENTITIES USING WAY-OUT FUNCTORS

As we just saw, it is often quite straightforward to prove that a given morphism $\eta\colon F \to G$ of functors on $\mathrm{D}_{\mathrm{qc}}(X)$ is an isomorphism on $\mathbf{Perf}(X)$. Fact 3.18 then gives us a way to bootstrap to an isomorphism on all of $\mathrm{D}_{\mathrm{qc}}(X)$ – assuming that F and G respect coproducts. Unfortunately, not every functor preserves coproducts.

Fortunately, there is a second way to bootstrap at least to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ (and this is the category we are all here for anyways). We start with a definition.

Definition 3.24. A functor $F\colon \mathrm{D}(\mathbf{A}) \to \mathrm{D}(\mathbf{B})$ between derived categories of abelian categories $\mathbf{A}$ and $\mathbf{B}$ is called *way-out left* if for any integer $n_1 \in \mathbb{Z}$ there exists an integer $n_2 \in \mathbb{Z}$ such that whenever X is in $\mathrm{D}^{\leq n_2}(\mathbf{A})$, then $F(X)$ is in $\mathrm{D}^{\leq n_1}(\mathbf{B})$. Analogously, one defines *way out right* with reversed inequalities, and reverses the inequality for n_2 for contravariant functors.

Here $\mathrm{D}^{\leq n}(\mathbf{A})$ consists of all those complexes $X \in \mathrm{D}(\mathbf{A})$ with $H^i(X) = 0$ for all $i > n$, and similarly one defines $\mathrm{D}^{\geq n}(\mathbf{A})$.

Fact 3.25 (Lemma on Way-Out Functors [H1, Proposition I.7.1])**.** *Let $\eta\colon F \to G$ be a natural transformation of functors $\mathrm{D}^-(\mathbf{A}) \to \mathrm{D}(\mathbf{B})$. Let P be a subset of the objects of $\mathbf{A}$ such that every object of $\mathbf{A}$ admits an surjection from an object of P. Assume that F and G are way-out left and $\eta(X)$ is an isomorphism for every $X \in P$. Then $\eta(X)$ is an isomorphism for every $X \in \mathrm{D}^-(\mathbf{A})$.*

A dual statement holds for way-out left functors, and there is a version for contravariant functors. As a demonstration of how to use this fact, let us prove the following useful identity.

Lemma 3.26. *Let X be a separated noetherian scheme and $\mathscr{F}, \mathscr{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. Assume that $\mathscr{H}\!om(-,\mathscr{F})$ and $\mathscr{H}\!om(-,\mathscr{G})$ take $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ to itself. Then for any $\mathscr{E} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ there exists a canonical isomorphism*

$$\mathscr{E} \otimes \mathscr{H}\!om(\mathscr{F},\mathscr{G}) \xrightarrow{\sim} \mathscr{H}\!om(\mathscr{H}\!om(\mathscr{E},\mathscr{F}), \mathscr{G}).$$

Proof. We will prove the statement under the additional assumption that $\mathscr{F}$ and $\mathscr{G}$ have finite injective dimension (this is the traditional way the statement is made). For a general proof without see [N2, Lemma 2.9].

The counit of tensor-Hom adjunction gives a map

$$\mathscr{E} \otimes \mathscr{H}\!om(\mathscr{E},\mathscr{F}) \to \mathscr{F}.$$

Applying this map twice, we obtain a map

$$\mathscr{E} \otimes \mathscr{H}\!om(\mathscr{E},\mathscr{F}) \otimes \mathscr{H}\!om(\mathscr{F},\mathscr{G}) \to \mathscr{F} \otimes \mathscr{H}\!om(\mathscr{F},\mathscr{G}) \to \mathscr{G}.$$

By adjunction we thus obtain a map

$$\mathscr{E} \otimes \mathscr{H}om(\mathscr{F}, \mathscr{G}) \to \mathscr{H}om(\mathscr{H}om(\mathscr{E}, \mathscr{F}), \mathscr{G}). \tag{1}$$

Checking that this is an isomorphism is local problem, so we can assume that X is affine. By the Lemma on Way-Out Functors together with [H1, Proposition II.7.20(ii)] it suffices to check that (1) is an isomorphism for $\mathscr{E} = \mathscr{O}_X^n$. Everything commutes with finite sums, so we further reduce to $\mathscr{E} = \mathscr{O}_X$, where the statement is trivial. □

3.5. FOURIER–MUKAI FUNCTORS

Given two schemes X and Y, we want to construct functors $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$ (or between the corresponding categories of quasi-coherent sheaves). For this we will take some inspiration from the Fourier transform in analysis. Roughly speaking, the Fourier transform associates to a function f on the real line the function

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

on the circle $S^1 = [0, 1]/(0 \sim 1)$. Decomposing this, the Fourier transform consists of three steps:

(i) Pulling f back to a function $\tilde{f}$ on $\mathbb{R} \times S^1$: $\tilde{f}(x, \xi) = f(x)$.

(ii) Multiplying $\tilde{f}$ by the *integral kernel* $e^{2\pi i x \xi}$.

(iii) Pushing the result forward to S^1, i.e. taking the integral over the first variable.

We can emulate this procedure in geometry step by step.

Definition 3.27. Let X and Y be separated noetherian schemes and $\mathscr{E} \in \mathrm{D}_{\mathrm{qc}}(X \times Y)$. Write $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ for the two projections. The *Fourier–Mukai functor* associated to $\mathscr{E}$ is the functor

$$\Phi_{\mathscr{E}} : \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y), \quad \Phi_{\mathscr{E}}(-) = q_*(\mathscr{E} \otimes p^*(-)).$$

We call $\mathscr{E}$ the *(Fourier–Mukai) kernel* of $\Phi_{\mathscr{E}}$.

Exercise 3.28. Consider the following “baby analogue”: X and Y are each just finitely many (closed) points, and instead of $\mathrm{D}_{\mathrm{qc}}(X)$ consider the functions $\mathrm{Funct}(X)$ from X to a fixed base field k. Describe “FM functions” $\mathrm{Funct}(X) \to \mathrm{Funct}(\mathrm{Y})$ (in this context pushforward is summation over fibers).

If X and Y are smooth and proper and $\mathscr{E} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X \times Y)$, then one is guaranteed that $\Phi_{\mathscr{E}}$ restricts to a functor $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$. Thus in the theory of Fourier–Mukai functors one often restricts to these assumptions.

Example 3.29. Consider the case $X = Y$ and let $\Delta\colon X \to X \times X$ be the diagonal. Let $\mathscr{E} \in \mathrm{D}_{\mathrm{qc}}(X)$. Then,

$$\begin{aligned}
\Phi_{\Delta_*\mathscr{E}}(\mathscr{F}) &= q_*(\Delta_*\mathscr{E} \otimes p^*\mathscr{F}) \\
&= q_*\Delta_*(\mathscr{E} \otimes \Delta^*p^*\mathscr{F}) && \text{(projection formula)} \\
&= (q \circ \Delta)_*(\mathscr{E} \otimes (p \circ \Delta)^*\mathscr{F}) \\
&= \mathscr{E} \otimes \mathscr{F} && (p \circ \Delta = q \circ \Delta = \mathrm{Id.})
\end{aligned}$$

Thus $\Phi_{\Delta_*\mathscr{E}}$ is tensoring by $\mathscr{E}$. In particular, $\Phi_{\Delta_*\mathscr{O}_X} = \mathrm{Id}$. ○

Exercise 3.30. Consider a correspondence $X \xleftarrow{f} Z \xrightarrow{g} Y$. What is the Fourier–Mukai kernel for g_*f^*?

Example 3.31. The most famous example of a Fourier–Mukai functor – and the reason for the name – is a theorem of Mukai [M2]. Let A be an abelian variety with dual $\hat{A}$ and let $\mathscr{P}$ be the Poincaré bundle on $A \times \hat{A}$, i.e., the unique bundle with the following two properties:

(i) If $\alpha \in \hat{A}$ corresponds to a line bundle $\mathscr{L} \in \mathrm{Pic}(A)$ on A, the $\mathscr{P}|_{A\times\{\alpha\}}$ is isomorphic to $\mathscr{L}$.

(ii) The restriction $\mathscr{P}|_{\{e\}\times\hat{A}}$ is trivial.

Then $\Phi_{\mathscr{P}}$ is an equivalence of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(A)$ with $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\hat{A})$. ○

Fourier–Mukai functors give us a large amount of functors between categories of sheaves, all of which respect coproducts. Moreover, these functors are quite straightforward to control.

Lemma 3.32. *Let X, Y and Z be smooth and proper schemes. Consider $\mathscr{E}_1 \in \mathrm{D}_{\mathrm{qc}}(X \times Y)$ and $\mathscr{E}_2 \in \mathrm{D}_{\mathrm{qc}}(Y \times Z)$. Then the FM kernel for the composition $\Phi_{\mathscr{E}_2} \circ \Phi_{\mathscr{E}_1}$ is given by*

$$\mathrm{pr}_{13,*}(\mathrm{pr}_{12}^*\,\mathscr{E}_1 \otimes \mathrm{pr}_{23}^*\,\mathscr{E}_2),$$

where pr_{ij}, $1 \leq i < j \leq 3$, is the projection from $X \times Y \times Z$ to the ith and jth factor.

$$\begin{tikzcd}
 & & X \times Z & & \\
 & & X \times Y \times Z \arrow[u, "\mathrm{pr}_{13}"] \arrow[dl, "\mathrm{pr}_{12}"] \arrow[dr, "\mathrm{pr}_{23}"] & & \\
 & X \times Y \arrow[dl] \arrow[dr] & & Y \times Z \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

Exercise 3.33. In the setup of Exercise 3.28, explicitly compute the composition of two kernels.

The proof of Lemma 3.32 is fairly straightforward, but a notational nightmare. One can find it written out it for example in [H3, Proposition 5.10]. The assumption that X, Y and Z are smooth and proper is only used to apply flat base change in one step of the proof. It is of course way stronger than what is really needed. Alternatively, one could switch to derived algebraic geometry, where base change holds under minimal assumptions.

It is also quite easy to give the kernels for left and right adjoint functors to $\Phi_{\mathscr{E}}$. Morally speaking they should be dual to $\mathscr{E}$ in an appropriate sense. Thus we will defer the discussion of adjoints to FM functors until the next section, where we will discuss duality questions in detail.

How large is the class of Fourier–Mukai functors? Morally speaking they are everything: any functor between the derived categories of reasonable schemes you will ever encounter is equivalent to a FM functor. The following theorem makes this precise.

Theorem 3.34 ([T, Theorem 8.9])**.** *Let X and Y be quasi-compact and quasi-separated schemes. The dg-category of coproduct preserving dg-functors between the dg-enhancements of $\mathrm{D}_{\mathrm{qc}}(X)$ and $\mathrm{D}_{\mathrm{qc}}(Y)$ is equivalent to the dg-enhancement of $\mathrm{D}_{\mathrm{qc}}(X \times Y)$.*

Remark 3.35. The above theorem talks about dg-functors. Any "reasonable" functor of derived categories you will encounter will have an enhancement to a dg functor between the corresponding enhanced categories. There are however some "unreasonable" (but still coproduct preserving!) functors between derived categories of coherent sheaves around, that are not equivalent to an FM functor, though the first such example was only found very recently [RVN]! There is also the following positive result.

Theorem 3.36 ([O2, Theorem 3.2.1])**.** *Let X and Y be smooth and projective over a field and assume that $F\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$ is fully faithful. Then F is equivalent to an FM functor.*

As usual, when one switches to derived algebraic geometry, various restrictions disappear. We refer to [BFN; BNP].

4. GROTHENDIECK DUALITY (A.K.A. GROTHENDIECK–SERRE DUALITY A.K.A. COHERENT DUALITY)

We have noted before that a complex in $\mathrm{D}_{\mathrm{qc}}(X)$ is dualizable (in the sense of monoidal categories) if and only if it is perfect. We also saw that in general there are more bounded coherent complexes than perfect one. Thus not every bounded complex of coherent sheaves is dualizable.

On the other hand it is quite reasonable to think of objects in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ as "finite", and finite objects *should* be dualizable. Our goal is thus to construct a duality functor $\mathbb{D}\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)^{\mathrm{op}}$ such that $\mathbb{D}^2 = \mathrm{Id}$. By the discussion above, the naive definition $\mathbb{D} = \mathscr{H}\!om(-, \mathscr{O}_X)$ will in general not have good properties (or even give an endofunctor of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$).

To motivate out approach, let as consider a proper variety X over a field k. Let $p\colon X \to \operatorname{Spec} k$ be the canonical map. Then we have for any $\mathscr{F} \in \mathrm{D}_{\mathrm{qc}}(X)$:

$$H^*(\mathscr{F})^\vee = (p_*\mathscr{F})^\vee = \mathrm{Hom}_k(p_*\mathscr{F}, k).$$

Assume that p_* has a right adjoint functor $p^!$. Then the above should be equal to $p_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, p^!k)$. It thus seems reasonable to define $\mathbb{D} = \mathcal{H}om(-, p^!k)$. Thus we have to look for the existence of a right adjoint to p_*. (Let us note that the above does not quite work if X is not proper. We will discuss the needed modifications in Section 4.2 below.)

4.1. A NON-STANDARD FUNCTOR

Being algebraic geometers, we want everything to be relative. Thus we want to find a right adjoint $f^\times$ of f_* any morphism $f\colon X \to Y$ of schemes, not just for the structure morphism.

Example 4.1. For a closed immersion $i\colon Z \hookrightarrow X$ one easily sees by tensor-hom adjunction that $i^\times = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z, -)$. On the other hand if $j\colon U \hookrightarrow X$ is an open immersion, then $j^\times$ is very non-obvious. ○

Classically there are two approaches to this problem: Via abstract nonsense, or by constructing dualizing complexes. We will take the first approach as it ties in quite nicely with the material of the previous section. Everything hinges on the following theorem:

Theorem 4.2 (Brown Representability [N1, Theorem 3.1]). *Let* $\mathbf{T}$ *be a compactly generated triangulated category. Assume that* $H\colon \mathbf{T}^{\mathrm{op}} \to \mathbf{Ab}$ *is a homological (contravariant) functor, i.e. takes distinguished triangles to short exact sequences. Suppose further that for all (small) coproducts in* $\mathbf{T}$ *the natural map*

$$H\Big(\coprod_\lambda X_\lambda\Big) \to \prod_\lambda H(X_\lambda)$$

is an isomorphism. Then H *is representable, i.e. there exists an object* $X \in \mathbf{T}$ *such that* $H \cong \mathrm{Hom}_{\mathbf{T}}(-, X)$.

We will not prove this theorem, but let us note that the proof, while non-obvious, does not require any fancy technology.

Corollary 4.3 (Adjoint Functor Theorem). *Assume* $F\colon \mathbf{S} \to \mathbf{T}$ *is an exact functor of triangulated categories. Assume further that* $\mathbf{S}$ *is compactly generated and* F *respects coproducts. Then* F *has a right adjoint* $G\colon \mathbf{T} \to \mathbf{S}$.

Proof. Fix any object $t \in \mathbf{T}$ and consider the functor

$$\mathbf{S} \to \mathbf{Ab}\colon s \mapsto \mathrm{Hom}_{\mathbf{T}}(F(s), t).$$

This functor is homological and takes coproducts to products. Hence it is representable by an object $G(t) \in \mathbf{S}$:

$$\mathrm{Hom}_{\mathbf{T}}(F(s), t) = \mathrm{Hom}_{\mathbf{S}}(s, G(t)).$$

If we have a map $t_1 \to t_2$ in $\mathbf{T}$, we obtain a natural transformation of the corresponding functors above, and hence a map between the representing objects $G(t_1) \to G(t_2)$. Thus G extends to a functor right adjoint to F. □

We already know that f_* preserves coproducts (Lemma 3.11). Thus we immediately obtain our desired functor.

Corollary 4.4. *Let* $f\colon X \to Y$ *be a separated morphism of quasi-compact separated schemes. Then* $f_*\colon \mathrm{D}_{\mathrm{qc}}(X) \to \mathrm{D}_{\mathrm{qc}}(Y)$ *has a right adjoint, which we will denote by* $f^\times$.

4.2. THE EXCEPTIONAL PULLBACK

It turns out that if $j\colon U \to X$ is an open immersion, then $j^\times$ is really hard to compute. (To be honest, at this point we do not know how to compute $f^\times$ for anything other than a closed immersion. We will get to that in a bit.) On the other hand if $\mathbb{D}_X$ is a duality functor on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ and $\mathbb{D}_U$ is one on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(U)$, then one should have $j^* \circ \mathbb{D}_X = \mathbb{D}_U \circ j^*$. Thus one usually does not work with $f^\times$ directly.

Recall that if $f\colon X \to Y$ is a separated morphism of finite type, then there exists a factorization of f as $X \overset{j}{\hookrightarrow} Z \xrightarrow{g} Y$ such that j is open and g is proper (Nagata compactification).

Definition 4.5. Let $f\colon X \to Y$ be a separated morphism of finite type. We define the exceptional pullback $f^!$ as $j^*g^\times$.

One can show that $f^!$ is independent of the factorization up to canonical isomorphism, though this isn't trivial. We refer to [N3, footnote on page 38] for a list of references.

As far as computations go, this has also made our lives easier as we now only need to consider $f^\times$ for proper f.

4.3. HOW TO COMPUTE THIS?

So far everything we did was purely abstract. We only know how to compute $f^\times$ when f is a closed immersion.

Since we are mainly interested in $f^!$, we only need to understand $f^\times$ when f is proper. Let us however start with an arbitrary morphism $f\colon X \to Y$ of noetherian separated schemes. The projection formula and adjunction produce a morphism

$$f_*(f^*\mathscr{F} \otimes f^\times\mathscr{O}_Y) \xrightarrow{\sim} \mathscr{F} \otimes f_*f^\times\mathscr{O}_Y \to \mathscr{F} \otimes O_Y = \mathscr{F}.$$

Applying $(f_*, f^\times)$-adjunction to this yields a morphism

$$\chi : f^*\mathscr{F} \otimes f^\times\mathscr{O}_Y \to f^\times\mathscr{F}.$$

Theorem 4.6. *The morphism χ is an isomorphism if and only if f is proper and of finite Tor-dimension.*

Here we call f of finite tor dimension if there exists an integer m such that $H^i(f^*\mathscr{F}) = 0$ for all $i < m$ and $\mathscr{F} \in \mathbf{QCoh}(Y)$. The upshot of Theorem 4.6 is that it suffices to compute $f^\times\mathscr{O}_Y$.

To prove the theorem, we will use a similar approach to the proof of the projection formula: We will first show it for perfect complexes and then appeal to continuity. To show that $f^\times$ preserves coproducts, we will use the following lemma.

Lemma 4.7. *Let $F\colon \mathbf{S} \to \mathbf{T}$ be an exact functor of compactly generated triangulated categories with a right adjoint G. Then G respects coproducts if and only if F respects compact objects.*

Proof. Let C be a generating set of $\mathbf{S}$ consisting of compact objects. Assume that F respects compactness. Then for any $c \in C$,

$$\begin{aligned}\mathrm{Hom}_{\mathbf{S}}(c, G(\coprod x_\alpha)) &= \mathrm{Hom}_{\mathbf{T}}(F(c), \coprod x_\alpha)\\ &= \coprod \mathrm{Hom}_{\mathbf{T}}(F(c), x_\alpha)\\ &= \coprod \mathrm{Hom}_{\mathbf{T}}(c, G(x_\alpha))\\ &= \mathrm{Hom}_{\mathbf{S}}(c, \coprod G(x_\alpha)).\end{aligned}$$

Let z be the cone of the natural map $\phi\colon \coprod G(x_\alpha) \to G(\coprod x_\alpha)$. It follows that $\mathrm{Hom}(c, z) = 0$ for all $c \in C$. Therefore $z = 0$ and ϕ is an isomorphism.

The other direction can be shown using the adjunction in a similar manner. □

Fact 4.8. *f_* respects compact objects (i.e., sends perfect complexes to perfect complexes) if and only if f is proper and of finite tor dimension.*

On direction of this fact is well known [STACKS, Tag 08EV], the other is [LN, Theorem 1.2].

Proof of Theorem 4.6. If $\mathscr{F}$ is perfect, we get the following chain of isomorphisms for any $\mathscr{H} \in \mathrm{D}_{\mathrm{qc}}(X)$:

$$\begin{aligned}\mathrm{Hom}(\mathscr{H}, f^*\mathscr{F} \otimes f^\times \mathscr{O}_Y) &\cong \mathrm{Hom}(\mathscr{H}, \mathscr{H}\!om(f^*\mathscr{F}^\vee, f^\times \mathscr{O}_Y))\\ &\cong \mathrm{Hom}(\mathscr{H} \otimes f^*\mathscr{F}^\vee, f^\times \mathscr{O}_Y)\\ &\cong \mathrm{Hom}(f_*(\mathscr{H} \otimes f^*\mathscr{F}^\vee), \mathscr{O}_Y)\\ &\cong \mathrm{Hom}(f_*\mathscr{H} \otimes \mathscr{F}^\vee, \mathscr{O}_Y)\\ &\cong \mathrm{Hom}(f_*\mathscr{H}, \mathscr{F}) \cong \mathrm{Hom}(\mathscr{H}, f^\times \mathscr{F}).\end{aligned}$$

By Yoneda it follows that χ is an isomorphism for perfect $\mathscr{F}$. Since all functors respect coproducts the statement follows from Fact 3.18. □

Before we proceed, let us record the following useful consequences:

Corollary 4.9. *Assume that f is as in Theorem 4.6 (in fact the assumption of finite tor dimension is stronger than necessary). Then one has canonical isomorphisms*

$$\begin{aligned}\mathscr{H}\!om(f_*\mathscr{F}, \mathscr{G}) &\cong f_*\mathscr{H}\!om(\mathscr{F}, f^!\mathscr{G})\\ f^!\mathscr{H}\!om(\mathscr{G}_1, \mathscr{G}_2) &\cong \mathscr{H}\!om(f^*\mathscr{G}_1, f^!\mathscr{G}_2).\end{aligned}$$

Proof. The first isomorphism needs a bit of work, see for example [N1, Section 6]. The second isomorphism follows formally from the first: for any $\mathscr{F} \in \mathrm{D}_{\mathrm{qc}}(X)$ we have

$$\begin{aligned}\mathrm{Hom}(\mathscr{F}, f^!\mathscr{H}\!om(\mathscr{G}_1, \mathscr{G}_2)) &\cong \mathrm{Hom}(f_*\mathscr{F}, \mathscr{H}\!om(\mathscr{G}_1, \mathscr{G}_2))\\ &\cong \mathrm{Hom}(f_*\mathscr{F} \otimes \mathscr{G}_1, \mathscr{G}_2)\\ &\cong \mathrm{Hom}(f_*(\mathscr{F} \otimes f^*\mathscr{G}_1), \mathscr{G}_2)\\ &\cong \mathrm{Hom}(\mathscr{F} \otimes f^*\mathscr{G}_1, f^!\mathscr{G}_2)\\ &\cong \mathrm{Hom}(\mathscr{F}, \mathscr{H}\!om(f^*\mathscr{G}_1, f^!\mathscr{G}_2)).\end{aligned}$$

An application of the Yoneda Lemma yields the result. □

We already know how to compute $f^\times \mathcal{O}_Y$ if f is a closed immersion:

$$f^\times \mathcal{O}_Y = \mathcal{H}om(\mathcal{O}_X, \mathcal{O}_Y).$$

The other main case, where we can explicitly describe $f^\times \mathcal{O}_Y$ is the following major theorem.

Theorem 4.10. *Assume that f is smooth and proper of relative dimension n. Then $f^\times \mathcal{O}_Y = \Omega_f^n[n]$, where Ω_f^n is the top exterior power of the relative cotangent bundle $\Omega_X/f^*\Omega_Y$.*

There is now a (quite recent) relatively elementary proof of this theorem. However it is still quite involved and too long for the time we have. A fairly complete sketch of the proof with references can be found in [N3, Section 3].

Factoring an arbitrary morphism into open immersions, closed immersions and smooth proper morphisms, one obtains the following statement, tying everything back to the derived category of coherent sheaves. We refer to [N4, Lemma 3.12] for details.

Corollary 4.11. *Let $f\colon X \to Y$ be a finite-type, separated morphism of noetherian schemes. If f is of finite tor dimension, then $f^! \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y) \subseteq \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$.*

Assume now that X is a smooth and projective variety dimension n over a field k and let $p\colon X \to \mathrm{pt}$ be the structure morphism. We write $\omega_X = \Omega_X^n = \Omega_p^n$ for the canonical line bundle on X. If $\mathcal{E}$ is any vector bundle on X and i any integer we have by adjunction

$$\mathrm{Hom}_k(p_*(\mathcal{E}[i]), k) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}[i], \omega_X[n]).$$

Taking cohomology, we obtain the following classical statement

Corollary 4.12 (Serre Duality)**.** *Let X be a smooth and projective variety of dimension n over a field. Then for any vector bundle $\mathcal{E}$ there exists a canonical isomorphism*

$$H^i(\mathcal{E})^\vee = \mathrm{Ext}^{n-i}(\mathcal{E}, \omega_X).$$

We can now also formulate the promised adjoints to Fourier–Mukai functors.

Corollary 4.13. *Let X and Y be smooth and projective and let $\Phi_{\mathcal{E}}\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$ be the FM functor associated to $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X \times Y)$. Then $\Phi_{\mathcal{E}}$ has a left adjoint given by the kernel $\mathcal{H}om(\mathcal{E}, \pi_X^! \mathcal{O}_X)$, and a right adjoint given by the kernel $\mathcal{H}om(\mathcal{E}, \pi_Y^! \mathcal{O}_Y)$.*

This can be proven with a straightforward computation, cf. [H3, Proposition 5.9]. The same formulas hold in much larger generality, see [R].

4.4. DUALIZING COMPLEXES

It is now time to return to our original goal of obtaining an autoduality of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. Let us start with a general definition of the kind of object we are looking for.

Definition 4.14. An object $\mathcal{D} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is called a *dualizing complex* if $\mathcal{H}om(-, \mathcal{D})$ induces an equivalence $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)^{\mathrm{op}}$.

Remark 4.15. Historically [H1], one additionally supposes that $\mathcal{D}$ has finite injective dimension. The reason for this is mainly that at that point the technology for working with unbounded derived categories had not yet been invented. Since we are already assuming this technology for $\mathrm{D}_{\mathrm{qc}}(X)$ above, we will continue to do so here.

The definition of a dualizing complex goes back all the way to Hartshorne's book [H1], based on Grothendieck's ideas. More recently van den Bergh introduced *rigid dualizing complexes* in order to obtain duality statements in a non-commutative context [B3]. These are also the correct formulation in the commutative setting, but we will not pursue them here.

In order to prove that any given complex is dualizing, we need a more succinct characterization. For this we note that $\mathcal{H}om(-,\mathcal{D})$ is its own left adjoint:

$$\begin{aligned}\mathrm{Hom}(\mathcal{E},\mathcal{H}om(\mathcal{F},\mathcal{D})) &\cong \mathrm{Hom}(\mathcal{E}\otimes\mathcal{F},\mathcal{D})\\ &\cong \mathrm{Hom}(\mathcal{F},\mathcal{H}om(\mathcal{E},\mathcal{D})) = \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)^{\mathrm{op}}}(\mathcal{H}om(\mathcal{E},\mathcal{D}),\mathcal{F}).\end{aligned}$$

The unit and counit of this adjunction are both given by the natural morphism

$$\mathcal{F}\to\mathcal{H}om(\mathcal{H}om(\mathcal{F},\mathcal{D}),\ \mathcal{D}).$$

Specializing to $\mathcal{F}=\mathcal{O}_X$, we obtain a natural morphism $\mathcal{O}_X\to\mathcal{H}om(\mathcal{D},\mathcal{D})$.

Lemma 4.16. *Let X be a separated, Noetherian scheme. The following are equivalent for $\mathcal{D}\in\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$.*

(i) *$\mathcal{D}$ is a dualizing complex.*

(ii) *$\mathcal{H}om(-,\mathcal{D})$ takes $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ to itself and for any $\mathcal{F}\in\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ the natural morphism $\mathcal{F}\to\mathcal{H}om(\mathcal{H}om(\mathcal{F},\mathcal{D}),\ \mathcal{D})$ is an isomorphism.*

(iii) *$\mathcal{H}om(-,\mathcal{D})$ takes $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ to itself and the natural morphism $\mathcal{O}_X\to\mathcal{H}om(\mathcal{D},\ \mathcal{D})$ is an isomorphism.*

Proof. As adjoint functors are equivalences if and only if both the unit and counit maps are isomorphisms, we see that (i) and (ii) are equivalent.

Clearly (ii) implies (iii). By Lemma 3.26, we have

$$\mathcal{H}om(\mathcal{H}om(\mathcal{F},\mathcal{D}),\ \mathcal{D})\cong\mathcal{F}\otimes\mathcal{H}om(\mathcal{D},\mathcal{D}).$$

Hence, in order to show that the map in (ii) is an isomorphism, it suffices to show that $\mathcal{O}_X\to\mathcal{H}om(\mathcal{D},\mathcal{D})$ is an isomorphism. ☐

Corollary 4.17. *Let $f\colon X\to Y$ be a morphism of separated noetherian schemes and assume that $\mathcal{D}$ is a dualizing complex on Y. Then $f^!\mathcal{D}$ is a dualizing complex on X.*

Sketch of proof. We will not show that $\mathcal{H}om(-,f^!\mathcal{D})$ takes $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ to itself – see [N2, Theorem 3.14] for details. To show that $\rho\colon\mathcal{O}_X\to\mathcal{H}om(f^!\mathcal{D},f^!\mathcal{D})$ is an isomorphism, it suffices to show that

$$k(x)\otimes\rho\colon k(x)\to k(x)\otimes\mathcal{H}om(f^!\mathcal{D},f^!\mathcal{D})\cong\mathcal{H}om(\mathcal{H}om(k(x),f^!\mathcal{D}),f^!\mathcal{D})$$

is an isomorphism for every closed point $x \in X$. The case of an open immersion being easy, we can assume that f is proper. But since everything is supported at a point, we may as well apply f_* and obtain

$$\begin{aligned} f_*k(x) \to f_*\mathcal{H}om(\mathcal{H}om(k(x),f^!\mathcal{D}),f^!\mathcal{D}) &\cong \mathcal{H}om(f_*\mathcal{H}om(k(x),f^!\mathcal{D}),\mathcal{D}) \\ &\cong \mathcal{H}om(\mathcal{H}om(f_*k(x),\mathcal{D}),\mathcal{D}). \end{aligned}$$

Thus the result follows from the assumption that $\mathcal{D}$ is a dualizing complex on Y. □

Fact 4.18 ([H1, Theorem V.3.1], [N2, Lemma 3.9])**.** *If X has a dualizing complex, then it is unique up to shift and tensoring with a line bundle.*

Remark 4.19. To make dualizing complexes unique, one should impose a rigidity condition, leading to *rigid dualizing complexes* introduced by van den Bergh [B3]. Working with rigid dualizing complexes also has the advantage that one can glue such complexes [YZ2]. We will satisfy ourselves with the following special case where a canonical choice exists without rigidity condition.

Definition 4.20. If X is a separated noetherian scheme over a field k and $p\colon X \to \operatorname{Spec} k$ the structure map, we set $\mathbb{dc}_X = p^!k$. We call $\mathbb{D}_X = \mathcal{H}om(-,\mathbb{dc}_X)$ the *dualizing functor* of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$.

From Corollary 4.9 we obtain a canonical isomorphism $f^* \cong \mathbb{D}_X f^! \mathbb{D}_Y$ for any (sufficiently well-behaved) $f\colon X \to Y$.

Example 4.21. As we have seen earlier, if X is smooth over k, then $\mathbb{dc}_X = \omega_X[\dim X]$. More generally a scheme is Gorenstein if and only if its dualizing complex is a shift of a line bundle [H1, Section V.9]. ○

Remark 4.22. As notes earlier, there are two main approaches to Grothendieck duality: defining dualizing complexes first, or defining the exceptional pullback first. The former approach was taken by Grothendieck and Hartshorne [H1] and developed into a modern theory by Yekutieli and Zhang [YZ1]. It has the advantage of being fairly concrete and amenable to computations (e.g., for proving Theorem 4.10), but has the drawback of needing to check lots of compatibilities for $f^!$ by hand. On the other hand, the latter approach – started by Deligne in the appendix to [H1], and then mainly developed by Lipman and Neeman, and taken by us here – makes it easy to obtain abstract properties of $f^!$, but Theorem 4.10 was only obtained very recently directly from this approach.

5. THE BONDAL–ORLOV RECONSTRUCTION THEOREM

We have been studying $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ and related categories for some time now, claiming that they are a useful invariant of X. An obvious question that arises is how much $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ actually remembers about X. It turns out that it remembers quite a lot. The aim of this section is to prove the following theorem in this direction.

Theorem 5.1 ([BO3, Theorem 2.5])**.** *Let X be a smooth irreducible projective variety over a field k with ample canonical or anticanonical sheaf. If $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is equivalent to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X')$ for some smooth variety X', the X is isomorphic to X'.*

Remark 5.2. A variety with ample anticanonical bundle is called *Fano*, while varieties with ample canonical bundle are examples of *varieties of general type* (a variety is of general type if its canoncial bundle is big).

To prove this theorem we need to be able pick out the canonical sheaf inside of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ without a-priori knowledge of X. We will do so by using the notion of a Serre functor.

Definition 5.3 ([BK])**.** Let $\mathbf{T}$ be a k-linear triangulated category with finite dimensional Hom spaces. An additive auto-equivalence S of $\mathbf{T}$ is called a *Serre functor* if there are given bi-functorial isomorphisms

$$\phi_{A,B} \colon \operatorname{Hom}_{\mathbf{T}}(A,B) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{T}}(B,S(A))^{\vee}.$$

Example 5.4. Let X be a smooth projective variety over k. Then by Grothendieck–Serre duality, $S = - \otimes \mathbb{dc}_X$ is a Serre functor on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \mathbf{Perf}(X)$:

$$\begin{aligned} \operatorname{Hom}(\mathscr{F},\mathscr{G})^{\vee} &= \operatorname{Hom}_k(p_*\mathscr{H}om(\mathscr{F},\mathscr{G}),k) \\ &= \operatorname{Hom}(\mathscr{H}om(\mathscr{F},\mathscr{G}),p^!k) \\ &= \operatorname{Hom}(\mathscr{F}^{\vee}\otimes\mathscr{G},\mathbb{dc}_X) \\ &= \operatorname{Hom}(\mathscr{G},\mathscr{F}\otimes\mathbb{dc}_X). \end{aligned} \qquad \bigcirc$$

Proposition 5.5. *A Serre functor commutes with any autoequivalence of* $\mathbf{T}$*, and hence in particular with the shift functor. Moreover it is an exact functor, i.e. takes distinguished triangles to distinguished triangles. If a category has a Serre functor, then it is unique up to natural isomorphism.*

Proof. The proofs of these statements can be found in [BO3, Section 1] and [BK, Section 3]. □

It follows that any smooth and projective variety X, given the triangulated category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ we always "know" the endofunctor $- \otimes \mathbb{dc}_X$.

Our next task for the proof of Theorem 5.1 is to find the set of closed points of X. Since we work with sheaves, we will do so by finding all skyscraper sheaves of closed points in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$.

Applying the Serre functor to such a skyscraper sheaf, i.e., tensoring it with a shift by $\dim X$ of the canonical line bundle of X, one gets the same skyscraper sheaf back, except for a shift by the dimension of X. We turn this into a definition:

Definition 5.6. Let $\mathbf{T}$ be a k-linear triangulated category with a Serre functor S. An object P of $\mathbf{T}$ is called a *point object* if the following conditions hold:

(i) $S(P) = P[s]$ for some integer s,

(ii) $\operatorname{Hom}(P,P[i]) = 0$ for $i < 0$,

(iii) $\operatorname{Hom}(P,P) = k(P)$ is a field extension of k.

The integer s is called the *codimension* of P. Since we assume that all Hom-spaces in $\mathbf{T}$ are finite, the field extension $k(P)$ of k is automatically finite.

Lemma 5.7. *Let X be as in Theorem 5.1. Then an object $P \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is a point object if and only if it is isomorphic to a shift of a skyscraper sheaf $\mathcal{O}_x$ of a closed point $x \in X$.*

Proof. One direction is obvious. Thus assume that P is a point object. Recall that $S \cong - \otimes \mathbb{dc}_X = - \otimes \omega_X[\dim X]$, where ω_X is the canonical line bundle.

Let $\mathcal{H}^i$ be the cohomology sheaves of P. Since only finitely many $\mathcal{H}^i$ are non-zero, (i) implies that $s = \dim X$ and $\mathcal{H}^i \otimes \omega_X = \mathcal{H}^i$. It follows that the Hilbert polynomial of $\mathcal{H}^i$ is constant. As the degree of the Hilbert polynomial is the dimension of the support of $\mathcal{H}^i$, each cohomology sheaf $\mathcal{H}^i$ is supported on a finite number of closed points of X.

If $\mathcal{H}^i$ was supported on more than one point, then $\mathrm{Hom}(\mathcal{H}^i, \mathcal{H}^i)$ would be decomposable, in contradiction to (iii). A spectral sequence argument, together with (ii) and (iii) then shows that only one $\mathcal{H}^i$ can be non-zero and it has to be the skyscraper sheaf at a closed point, see [BO3] for details. ☐

We observe that in particular $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ knows about the dimension of X as the codimension of any point object. It also follows that the same statement holds on X': Suppose there is a point object P in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X')$ that is not of the form $\mathcal{O}_{x'}[r]$. All point objects correspond to shifts of skyscraper sheaves on X. Thus $\mathrm{Hom}(P', P) = \mathrm{Hom}(P, P') = 0$ for any P' that is not isomorphic to P. This holds in particular for P' any shift of a skyscraper sheaf on X', forcing P to be zero.

Next we will find all line bundles in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. These will be helpful for recovering the topology of X.

Lemma 5.8. *Assume that X is a smooth projective variety satisfying the conclusion of Lemma 5.7. An object $L \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is a shift of a line bundle if and only if for every point object P there exists an integer s such that*

(i) $\mathrm{Hom}(L, P[s]) = k(P)$,

(ii) $\mathrm{Hom}(L, P[i]) = 0$ *for $i \neq s$.*

Proof. We already know that point objects are of the form $\mathcal{O}_x[r]$ for some closed point $x \in X$.

If L is a shift of a line bundle it clearly satisfies the two conditions. Conversely, suppose L satisfies the two conditions and let $\mathcal{H}^i$ be the cohomology sheaves of L. Write i_0 for the largest integer such that $\mathcal{H}^{i_0} \neq 0$ (note that by the first condition $L \neq 0$). For any closed point $x \in X$ consider the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p(\mathcal{H}^q, \mathcal{O}_x) \Rightarrow \mathrm{Ext}^{p-q}(L, \mathcal{O}_x).$$

If x is in the support of $\mathcal{H}^{i_0}$, then $\mathrm{Hom}(\mathcal{H}^{i_0}, \mathcal{O}_x) \neq 0$. By maximality of i_0, both $\mathrm{Hom}(\mathcal{H}^{i_0}, \mathcal{O}_x)$ and $\mathrm{Ext}^1(\mathcal{H}^{i_0}, \mathcal{O}_X)$ are left intact throughout the spectral sequence. It follows that necessarily

$$\mathrm{Hom}(\mathcal{H}^{i_0}, \mathcal{O}_x) = k(x)$$
$$\mathrm{Ext}^1(\mathcal{H}^{i_0}, \mathcal{O}_x) = 0.$$

Using the fact that a finite module M over a Noetherian local ring $(A, \mathfrak{m})$ is free if and only if $\mathrm{Ext}^1(M, A/\mathfrak{m}) = 0$, the second line implies that $\mathcal{H}^{i_0}$ is locally free. The first line that says that $\mathcal{H}^{i_0}$ is a line bundle.

Finally, a descending induction argument using the above spectral sequence together with the assumptions shows that all other $\mathcal{H}^i$ have to vanish. □

Proof of Theorem 5.1. So far we found *all shifts* of skyscraper sheaves and *all shifts* of line bundles. We use the following trick to get rid of the shifts:

Fix an object L_0 satisfying the conditions in Lemma 5.8[1]. Then the set of skyscraper sheaves (up to a *fixed* shift) is exactly those point objects P such that $\operatorname{Hom}(L_0, P) = k(P)$. Conversely, the set of line bundles (up to *the same* fixed shift) are those objects L as in Lemma 5.8 such that $\operatorname{Hom}(L, P) = k(P)$ for any skyscraper P. In particular the set of closed points of X is in bijection with isomorphism classes of point objects P such that $\operatorname{Hom}(L_0, P) = k(P)$.

It follows that X and X' have the same skyscraper sheaves and line bundles. In particular we obtain a bijection of the closed points of X with those of X'.

Next we want to reconstruct the topologies of X and X'. Suppose ω_X is ample. For any $s \in \Gamma(X, \omega_X^{\otimes n})$ set $X_s = \{x \in X : s \notin \mathfrak{m}_x \omega_{X,x}^{\otimes n}\}$. Then the sets X_s form a basis of the topology of X [STACKS, Tag 01PR].

To formulate this without having direct access to ω_X we fix a line bundle L_0 and set $L_i = S^i(L_0)[-i \dim X]$ (where S is the Serre functor). Then the sets $\operatorname{Hom}(L_i, L_j)$ certainly include all $\Gamma(X, \omega_X^{\otimes n})$ – as well as all $\Gamma(X, \omega_X^{\otimes -n})$ for the antiample case. For each $\alpha \in \operatorname{Hom}(L_i, L_j)$ let U_α be the set of all skyscraper sheaves P such that the induced map $\operatorname{Hom}(L_j, P) \to \operatorname{Hom}(L_i, P)$ is non-zero. The (open) sets U_α include all sets X_s from above and hence form a basis of the Zariski topology of X.

More generally, one can show that if we run this construction for all pairs of line bundles on X (resp. X'), the U_α always form a basis of the topology of X (resp. X'). As the classes of line bundles are the same on the two varieties, the set bijection we obtained above upgrades to a homeomorphism of topological spaces. But then the U_α obtained from just the L_i also form a basis of the topology of X'. Using the same characterization of ampleness as above, this then implies that $\omega_{X'}$ has the same (anti)ampleness property as ω_X (here we use that the Serre functor commutes with the given equivalence).

We have

$$\Gamma(X, \omega_X^{\otimes n}) = \operatorname{Hom}(\mathcal{O}_X, \omega_X^{\otimes n}) = \operatorname{Hom}(L_0, L_n) = \operatorname{Hom}(\mathcal{O}_{X'}, \omega_{X'}^{\otimes n}) = \Gamma(X, \omega_{X'}^{\otimes n}).$$

As both X and X' have an ample (or antiample) line bundle they are determined by the graded ring

$$\bigoplus_n \Gamma(X, \omega_X^{\otimes n}) = \bigoplus_n \Gamma(X', \omega_{X'}^{\otimes n})$$

and thus are isomorphic. □

6. UNDERSTANDING DERIVED CATEGORIES

To efficiently work with derived categories of coherent sheaves we need to understand their structure. In particular, we would like to compare them to categories we can understand, or at least break them up into simpler categories.

[1]Such an object is called "invertible" in the literature, but we have decided against overloading this word even more.

6.1. BEILINSON'S RESULT

Maybe the first of this type of result is due to Beilinson and concerns $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^n)$ [B2]. Here we always work over a fixed ground field k. As Beilinson's article is rather terse, we also refer to [H3, Section 8.3] and [C1]. It all hinges on the following resolution of the Fourier–Mukai kernel of the identity functor.

Lemma 6.1 (Beilinson's resolution of the diagonal)**.** *Set* $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \Omega^1_{\mathbb{P}^n}(1)$ *on* $\mathbb{P}^n \times \mathbb{P}^n$. *There exists an exact sequence*

$$0 \to \bigwedge^n \mathcal{E} \to \cdots \to \bigwedge^2 \mathcal{E} \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0, \tag{2}$$

where $\mathcal{O}_\Delta$ *is the structure sheaf of the diagonal in* $\mathbb{P}^n \times \mathbb{P}^n$.

Proof. We start by considering the Euler exact sequence [H2, Theorem II.8.13]

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

Let p and q be the two projections $\mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$. Then the Euler exact sequence induces a morphism

$$q^*\Omega^1_{\mathbb{P}^n}(1) \to q^*\mathcal{O}^{n+1}_{\mathbb{P}^n} \cong p^*\mathcal{O}^{n+1}_{\mathbb{P}^n} \to p^*\mathcal{O}_{\mathbb{P}^n}(1).$$

Tensoring with $p^*\mathcal{O}_{\mathbb{P}^n}(-1)$ we obtain the desired morphism

$$\epsilon : \mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \Omega^1_{\mathbb{P}^n}(1) = p^*\mathcal{O}_{\mathbb{P}^n}(-1) \otimes q^*\Omega^1_{\mathbb{P}^n}(1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}.$$

Explicitly, the fiber of $\mathcal{O}_{\mathbb{P}^n}(-1)$ at a point $l \in \mathbb{P}^n$ corresponds to the line l itself. On the other hand the fiber of $\Omega^1_{\mathbb{P}^n}(1)$ at l corresponds to linear maps $k^{n+1} \to k$ vanishing on l. Over a point (l_1, l_2) the map ϵ then sends a pair (v, ϕ) to $\phi(v)$. It is clear that this morphism vanishes if and only if $l_1 = l_2$, i.e. on the diagonal. We see that the cokernel of ϵ is $\mathcal{O}_\Delta$ (this can also be easily verified in coordinates, see [C1, Theorem III.1.1]).

The sequence (2) is then just the Koszul complex associated to ϵ. Explicitly the differential

$$\bigwedge^p \mathcal{E} \to \bigwedge^{p-1} \mathcal{E}$$

is given by

$$s_1 \wedge \cdots \wedge s_p \mapsto \sum_{j=1}^{p} (-1)^{j-1} \epsilon(s_j) s_1 \wedge \cdots \wedge \widehat{s_j} \wedge \cdots \wedge s_p.$$

Exactness can be tested locally, where it follows from standard results on the Koszul complex, see for example [E, Chapter 17]. □

Corollary 6.2. *For any integer a, the sheaves* $\mathcal{O}_{\mathbb{P}^n}(a-n), \dots, \mathcal{O}_{\mathbb{P}^n}(a)$ *generate* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^n)$ *as a triangulated category, i.e. the smallest full triangulated subcategory of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ *containing* $\mathcal{O}(a-n), \dots, \mathcal{O}(a)$ *is equivalent to* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ *itself.*

Proof. Since $- \otimes \mathcal{O}(-a)$ is an exact auto-equivalence of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^n)$, we can assume that $a = 0$. We also note that we have a natural isomorphism

$$\bigwedge^k (\mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \Omega^1_{\mathbb{P}^n}(1)) \cong \mathcal{O}_{\mathbb{P}^n}(-k) \boxtimes \Omega^k_{\mathbb{P}^n}(k).$$

As before write p and q for the two projections $\mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$. The resolution (2) of $\mathcal{O}_\Delta$ gives a resolution of the identity functor $\mathrm{Id}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^n)} = p_*(\mathcal{O}_\Delta \otimes q^*(-))$ by functors of the form[2]

$$\begin{aligned} p_*(p^*\mathcal{O}_{\mathbb{P}^n}(-k) \otimes q^*\Omega^k_{\mathbb{P}^n}(k) \otimes q^*(-)) &\cong \mathcal{O}_{\mathbb{P}^n}(-k) \otimes p_*(q^*(\Omega^k_{\mathbb{P}^n}(k) \otimes -)) \\ &\cong \mathcal{O}_{\mathbb{P}^n}(-k) \otimes_k \Gamma(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(k) \otimes -), \end{aligned}$$

where the first isomorphism follows from the projection formula (Theorem 3.23), and the second from flat base change (Fact 3.4). (As always, all functors – including the global sections functor Γ – are taken to be derived.) It thus suffices that the images of these functors are contained in the subcategory generated by $\mathcal{O}_{\mathbb{P}^n}(-n), \ldots, \mathcal{O}_{\mathbb{P}^n}$.

For any $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^n)$, its global sections are a complex of finite dimensional vector spaces. Since triangulated categories are by definition closed under finite sums, it follows that $\mathcal{O}_{\mathbb{P}^n}(-k) \otimes_k \Gamma(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(k) \otimes \mathcal{F})$ is contained in the triangulated subcategory generated by $\mathcal{O}_{\mathbb{P}^n}(-k)$. $\square$

We notice that the Ext's between the $\mathcal{O}(i)$ above are particularly nice [H2, Proposition II.5.13 and Theorem III.5.1]:

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(i),\ \mathcal{O}_{\mathbb{P}^n}(i)[\ell]) = H^\ell(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = \begin{cases} k & \text{if } \ell = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and for $a - n \le i < j \le a$ and any $\ell \in \mathbb{Z}$ we have

$$\mathrm{Hom}(\mathcal{O}(j), \mathcal{O}(i)[\ell]) = H^\ell(\mathbb{P}^n, \mathcal{O}(i-j)) = 0.$$

We turn this into a definition.

Definition 6.3. Let $\mathbf{T}$ be a k-linear triangulated category.

- An object $E \in T$ is *exceptional* if

$$\mathrm{Hom}(E, E[\ell]) = \begin{cases} k & \text{if } \ell = 0, \\ 0 & \text{otherwise.} \end{cases}$$

- A sequence $E_1, \ldots, E_n$ of objects in $\mathbf{T}$ is *exceptional* if each E_i is exceptional and additionally for $1 \le i < j \le n$ and any $\ell \in \mathbb{Z}$

$$\mathrm{Hom}(E_j, E_i[\ell]) = 0$$

[2] Here we play somewhat loose with category theory. The usual way to make this argument fit into the theory of triangulated categories is to split the sequence (2) into a distinguished triangles and proceed by induction. We refer to the given references for details.

- An exceptional sequence is *strong* if additionally for any $1 \leq i < j \leq n$ and $\ell \neq 0$

$$\operatorname{Hom}(E_i, E_j[\ell]) = 0.$$

- A sequence is *full* if it generates $\mathbf{T}$ as a triangulated category.

Corollary 6.4. *For any integer a the sequence of sheaves $\mathcal{O}_{\mathbb{P}^n}(a-n), \ldots, \mathcal{O}_{\mathbb{P}^n}(a)$ is a full strong exceptional sequence in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^n)$.*

Let V be an $n+1$ dimensional k-vector space. Then [H2, Proposition II.5.13] implies that the remaining non-vanishing Hom's are

$$\operatorname{Hom}(\mathcal{O}_{\mathbb{P}^n}(i),\ \mathcal{O}_{\mathbb{P}^n}(j)) = \operatorname{Sym}^{j-i} V.$$

Let $\mathbf{M}_S$ be the category whose objects are finite direct sums of $\operatorname{Sym}^\bullet V(-i)$, for $0 \leq i \leq n$ and morphisms are degree 0 morphisms of graded $\operatorname{Sym}^\bullet V$-modules. Here for a graded algebra $A^\bullet$, we write $A^\bullet(i)$ for the free one-dimensional graded $A^\bullet$-module with generator in degree $-i$. Write $\mathbf{K}_S$ for the homotopy category of bounded complexes over $\mathbf{M}$.

Corollary 6.5 ([B2])**.** *Sending* $\operatorname{Sym}^\bullet V(-i)$ *to* $\mathcal{O}(-i)$ *gives an equivalence of triangulated categories* $\mathbf{K}_S \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^n)$.

Remark 6.6. Dually, $\mathcal{O}_{\mathbb{P}^n}, \Omega^1_{\mathbb{P}^n}(1), \ldots, \Omega^n_{\mathbb{P}^n}(n)$ also forms a strong full exceptional sequence in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^n)$. Replacing $\operatorname{Sym}^\bullet V$ by $\bigwedge^\bullet V^\vee$, one defines categories $\mathbf{M}_\Lambda$ and $\mathbf{K}_\Lambda$, and obtains an equivalence $\mathbf{K}_\Lambda \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}^n)$.

6.2. TILTING

There is another, more general way that Beilinson's exceptional sequence can be used to define an equivalence of categories.

Definition 6.7. Let $\mathbf{T}$ be a k-linear triangulated category. An object $T \in \mathbf{T}$ is called *tilting* if

(i) The k-algebra $R = \operatorname{Hom}_{\mathbf{T}}(T, T)$ has finite global dimension,

(ii) $\operatorname{Hom}_{\mathbf{T}}(T, T[\ell]) = 0$ for $\ell \neq 0$,

(iii) T classically generates $\mathbf{T}$, i.e. the smallest triangulated subcategory of $\mathbf{T}$ which contains T and is closed under isomorphisms and taking direct summands is $\mathbf{T}$ itself.

Lemma 6.8 ([B5], see also [C2, Proposition 2.7])**.** *Let X be a smooth projective variety. If $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is full strong exceptional sequence in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, then $\mathcal{T} = \bigoplus \mathcal{E}_i$ is tilting.*

The only non-trivial part of this statement is that the endomorphism algebra of $\mathcal{T}$ has finite global dimension.

Example 6.9. $\bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(i)$ is a tilting sheaf on $\mathbb{P}^n$. Its endomorphism algebra is $R = \operatorname{Sym}^\bullet V/(\operatorname{Sym}^{n+1} V)$. We note that in the construction of $\mathbf{K}_S$ above we could replace $\operatorname{Sym}^\bullet V$ by R and obtain an equivalent category. ○

Theorem 6.10 ([B5; B1], see also [C2, Theorem 2.1])**.** *Let X be a smooth projective variety and $\mathscr{T}$ a tilting object in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. Then the functor*

$$R\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{T},-)\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \to \mathrm{D}^b_f(R^{\mathrm{op}})$$

is an equivalence with quasi-inverse $-\otimes_R \mathscr{T}$.

Sketch of proof. The point is that

$$R\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{T}, R\otimes_R \mathscr{T}) = R\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{T},\mathscr{T}) = \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{T},\mathscr{T}) = R.$$

From this it follows that $R\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{T}, -\otimes_R \mathscr{T})$ is the identity on $\mathrm{D}^b_f(R^{\mathrm{op}})$. Hence $-\otimes_R \mathscr{T}$ is fully faithful and identifies $\mathrm{D}^b_f(R^{\mathrm{op}})$ with the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ generated by $\mathscr{T}$. By assumption, this is all of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. □

Remark 6.11. In the context of Theorem 6.10, instead of requiring that $\mathscr{T}$ classically generates $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, it suffices to require that $\mathscr{T}$ weakly generates $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, i.e. that whenever $\operatorname{Hom}(\mathscr{T},\mathscr{F}[\ell]) = 0$ for all $\ell \in \mathbb{Z}$, then already $\mathscr{F} = 0$ (see [BB, Theorem 2.1.2] and the reference given there). In fact one has the following statement, which is a special form of the Barr-Beck–Lurie Theorem.

Theorem 6.12. *Let $\mathbf{C}$ be a cocomplete dg category with a single compact generator M and let R be the dg algebra $= \operatorname{End}_{\mathbf{C}}(M)^{\mathrm{op}}$. Then the assignment $N \mapsto \operatorname{Hom}_{\mathbf{C}}(M,N)$ defines an equivalence $\mathbf{C} \cong \mathrm{D}(R)$.*

6.3. SEMI-ORTHOGONAL DECOMPOSITIONS

If $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ has a full exceptional sequence, then by the constructions above, we get as good an understanding of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ as one can reasonably expect. Unfortunately, for the same reason having a full exceptional collection is a very strong condition. Thus we weaken the requirements to the following:

Definition 6.13. A *semi-orthogonal decomposition* (often shortened to *SOD*) of a triangulated category $\mathbf{T}$ is a collection $\mathbf{A}_1,\dots,\mathbf{A}_n$ of full triangulated subcategories of $\mathbf{T}$ such that

(i) $\operatorname{Hom}(\mathbf{A}_i,\mathbf{A}_j) = 0$ whenever $j < i$ (this means that for any $A \in \mathbf{A}_i$ and $B \in \mathbf{A}_j$ we have $\operatorname{Hom}_{\mathbf{T}}(A,B) = 0$);

(ii) the smallest triangulated subcategory of $\mathbf{T}$ containing $\mathbf{A}_1,\dots,\mathbf{A}_n$ coincides with $\mathbf{T}$.

In this case one writes

$$\mathbf{T} = \langle \mathbf{A}_1,\dots,\mathbf{A}_n\rangle.$$

In this context it is convenient to make the following definition.

Definition 6.14. Let $\mathbf{A}$ be a full triangulated subcategory of $\mathbf{T}$.

(i) The *left orthogonal* of $\mathbf{A}$ is

$$^{\perp}\mathbf{A} = \{T \in \mathbf{T} : \operatorname{Hom}(T,A) = 0 \text{ for all } A \in \mathbf{A}\}.$$

(ii) The *right orthogonal* of $\mathbf{A}$ is

$$\mathbf{A}^{\perp} = \{T \in \mathbf{T} : \mathrm{Hom}(A, T) = 0 \text{ for all } A \in \mathbf{A}\}.$$

Thus the condition (i) of the definition of a semi-orthogonal decomposition can be written as $\mathbf{A}_j \subset \mathbf{A}_i^{\perp}$ for $j < i$.

Lemma 6.15. *Let* $\mathbf{A}$ *be a full triangulated subcategory of* $\mathbf{T}$. *Then* $\mathbf{A}^{\perp}$ *and* ${}^{\perp}\mathbf{A}$ *are triangulated.*

Proof. We will prove the statement for $\mathbf{A}^{\perp}$. The proof for the other side is completely analogous.

If $X \in \mathbf{A}^{\perp}$ then $\mathrm{Hom}(A, X[1]) = \mathrm{Hom}(A[-1], X) = 0$ for all $A \in \mathbf{A}$. Thus $\mathbf{A}^{\perp}$ closed under shifts.

Further, suppose $X_1 \to X_2 \to Z$ is a distinguished triangle with X_1 and X_2 in $\mathbf{A}^{\perp}$. For any $A \in \mathbf{A}$ we can apply $\mathrm{Hom}_T(A, -)$ to this triangle and obtain a long exact sequence

$$\cdots \to \mathrm{Hom}(A, X_1) \to \mathrm{Hom}(A, X_2) \to \mathrm{Hom}(A, Z) \to \mathrm{Hom}(A, X_1[1]) \to \cdots .$$

As $\mathrm{Hom}(A, X_2) = \mathrm{Hom}(A, X_1[1]) = 0$ also $\mathrm{Hom}(A, Z) = 0$. Thus $\mathbf{A}^{\perp}$ is closed under forming cones. □

Example 6.16. If $E_1, \dots, E_n$ is an exceptional sequence in $\mathbf{T}$, then

$$\mathbf{T} = \langle \mathbf{A}, E_1, \dots, E_n \rangle \quad \text{with} \quad \mathbf{A} = \langle E_1, \dots, E_n \rangle^{\perp}.$$

Here we abuse notation and also write E_n for the triangulated subcategory $\langle E_i \rangle$ generated by E_i. We note that this subcategory is equivalent to the bounded derived category of finite dimensional k-vector spaces. ○

In general one wants the categories forming an SOD to be of slightly special type:

Definition 6.17. A full triangulated subcategory $\mathbf{A}$ of $\mathbf{T}$ is called *admissible* if the inclusion functor $\mathbf{A} \to \mathbf{T}$ has both a right and left adjoint.

Exercise 6.18. If $\mathbf{A}$ is an admissible subcategory of $\mathbf{T}$, then

$$\langle \mathbf{A}, {}^{\perp}\mathbf{A} \rangle \quad \text{and} \quad \langle \mathbf{A}^{\perp}, \mathbf{A} \rangle$$

are semi-orthogonal decompositions of $\mathbf{T}$.

Remark 6.19. Some authors call $\mathbf{A}$ admissible if the inclusion functor only has a right adjoint. In this case one only gets the SOD $\langle \mathbf{A}^{\perp}, \mathbf{A} \rangle$

A common way to obtain semi-orthogonal decompositions is by applying the following observation.

Corollary 6.20. *Let* X *and* Y *be smooth projective varieties over a field* k. *If* $F\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$ *is fully faithful, then*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y) = \langle \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)^{\perp}, \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \rangle = \langle \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X), {}^{\perp}\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \rangle,$$

where we identify $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ *with its essential image under* F.

Proof. By Theorem 3.36, F is equivalent to a Fourier–Mukai functor. Thus by Corollary 4.13 F has left and right adjoints. Hence F identifies $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ with an admissible subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)$. □

To effectively use Corollary 6.20, one of course needs a good criterion for testing fully-faithfulness. The most famous example of such a criterion is [BO1, Theorem 1.1], which reduces the question to checking that the images of skyscraper sheaves behave correctly.

Semi-orthogonal decompositions of the derived category of coherent sheaves have now been given for many varieties. [K4] gives a partial – and by now quite out-dated – list. Let us state two early results. The first one is a relative version of Beilinson's exceptional sequence (Theorem 6.4).

Theorem 6.21 ([O1, Theorem 2.6]). *Let X be a scheme and E a vector bundle of rank r on X with projectivization $p\colon \mathbb{P}(E) \to X$. Let $\mathcal{O}_{\mathbb{P}(E)/X}(1)$ be the Grothendieck line bundle on $\mathbb{P}(E)$. Then for any integer i the functor*

$$\begin{aligned} \Phi_i\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) &\to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}(E)) \\ \mathcal{F} &\mapsto f^*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(E)/X}(i) \end{aligned}$$

is fully faithful and there is a semi-orthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathbb{P}(E)) = \big\langle \Phi_{-r+1}(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)), \dots, \Phi_0(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X))\big\rangle.$$

For the next theorem let X be a smooth projective variety and $Y \subseteq X$ a locally complete intersection subvariety of codimension c. Let $f\colon \tilde{X} \to X$ be the blow-up of X along Y and let $D \subset \tilde{X}$ be the exceptional divisor. Let $i\colon D \hookrightarrow \tilde{X}$ be the embedding and $p\colon D \to Y$ be the restriction of f. We note that D is the projectivization of the normal bundle of Y in X.

Theorem 6.22 ([O1, Theorem 4.3]). *The functor $f^*\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\tilde{X})$ as well as the functors*

$$\begin{aligned} \Psi_i\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y) &\to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\tilde{X}) \\ \mathcal{F} &\mapsto i_*(p^*\mathcal{F} \otimes \mathcal{O}_{D/Y}(i)) \end{aligned}$$

are fully faithful and there exists a semi-orthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\tilde{X}) = \big\langle f^*\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\tilde{X}), \Psi_0(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y)), \dots, \Psi_{c-2}(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y))\big\rangle.$$

Finally let us mention that semi-orthogonal decompositions tend to behave well under base-change. Of course for this to be possible the SOD cannot be completely arbitrary.

Definition 6.23. Let $f\colon X \to S$ be a morphism of schemes. A semi-orthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \langle \mathbf{A}_1, \dots, \mathbf{A}_m\rangle$$

is called *S-linear* if each of the subcategories $\mathbf{A}_i$ is closed under tensoring with objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(S)$, i.e. for each $\mathcal{A} \in \mathbf{A}_i$ and $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(S)$ one has $\mathcal{A} \otimes f^*\mathcal{F} \in \mathbf{A}_i$.

Both of the above semi-orthogonal decompositions are X-linear.

Theorem 6.24 ([K3, Theorem 5.6])**.** *Let X over S be a variety with an S-linear semi-orthogonal decomposition*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \langle \mathbf{A}_1, \dots, \mathbf{A}_m \rangle$$

consisting of admissible subcategories. Let $T \to S$ be a base-change with projection $\pi \colon X \times_S T \to X$. Then, under some technical assumptions, there exists a T-linear semi-orthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X \times_S T) = \langle \mathbf{A}_{1T}, \dots, \mathbf{A}_{mT} \rangle$$

such that $\pi^*\mathbf{A}_i \subseteq \mathbf{A}_{iT}$ and $\pi_*(A) \in \mathbf{A}_i$ for any $A \in \mathbf{A}_{iT}$ with proper support over X.

A very important source of – and application of – semi-orthogonal decompositions of derived categories is *homological projective duality* [K2]. We refer to the overview article [K4] for an outline, as well as many examples.

7. SINGULAR SUPPORT

As we showed in Theorem 3.14, the categories $\mathbf{Perf}(X)$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ agree if and only if X is smooth. Thus the *category of singularities* $\mathrm{D}_{\mathrm{sg}}(X) = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \big/ \mathbf{Perf}(X)$ is a measure of the singularities of X. The aim of this section is then to refine this observation and obtain a measure of how far an element of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is from being perfect.

7.1. KOSZUL DUALITY

As a motivation – and an observation of independent interest – we will consider a special form of Koszul duality from a geometric point of view. For this let us consider the fiber product $G_{\mathrm{pt}/\mathbb{A}^n} = \mathrm{pt} \times_{\mathbb{A}^n} \mathrm{pt}$. As an ordinary scheme $G_{\mathrm{pt}/\mathbb{A}^n}$ would of course just be a point again.

More interesting in this context is the point of view of derived algebraic geometry. In this setting $G_{\mathrm{pt}/asn}$ is the affine scheme corresponding to the derived tensor product $k \otimes_{k[x_1,\dots,x_n]} k$. To compute this product, we have to resolve k by free $R = k[x_1, \dots, x_n]$-modules. The standard way to do this is via the Koszul resolution

$$\bigwedge^n R^n \to \cdots \to \bigwedge^2 R^n \to R^n \to R \to k,$$

with differential

$$\begin{aligned} \bigwedge^p R^n &\to \bigwedge^{p-1} R^{n-1} \\ r_1 \wedge \cdots \wedge r_p &\mapsto \sum_{j=1}^{p} (-1)^j x_j r_1 \wedge \cdots \wedge \widehat{r_j} \wedge \cdots \wedge r_p. \end{aligned}$$

Thus $k \otimes_{k[x_1,\dots,x_n]} k$ is given by the complex

$$\bigwedge^n k^n \xrightarrow{0} \cdots \xrightarrow{0} \bigwedge^2 k^n \xrightarrow{0} k^n \xrightarrow{0} k.$$

This is just the exterior algebra on k^n with appropriate shifts. Using the usual convention that variables in odd degrees anti-commute, we can also write this as

$$k \otimes_{k[x_1,\dots,x_n]} k \cong \mathrm{Sym}(k^n[1]) \cong k[\lambda_1, \dots, \lambda_n],$$

with $\deg \lambda_i = -1$.

In derived algebraic geometry one interprets $G_{\mathrm{pt}/\mathbb{A}^n} = \operatorname{Spec} k[\lambda_1, \dots, \lambda_n]$ as the derived scheme with just a single point, but structure sheaf the dg-algebra $\Lambda = k[\lambda_1, \dots, \lambda_n]$ (with zero differential). The category of (quasi)coherent sheaves on $G_{\mathrm{pt}/\mathbb{A}^n}$ is just the category of (finitely generated) Λ-modules. The category of these module is far from trivial, but with the single point we are not even able to record any interesting support for coherent sheaves.

The idea is then to apply a variant of the tilting procedure of Section 6.2. For this we note that the module k is a weak generator of $\mathrm{D}_{\mathrm{qc}}(G_{\mathrm{pt}/\mathbb{A}^n}) = \mathrm{D}(\mathbf{Mod}(\Lambda))$. Let us remark however that it is not compact or perfect (i.e. quasi-isomorphic to a finite complex of free Λ-modules). In fact we have the following observation.

Lemma 7.1. *There is an isomorphism of dg-algebras*

$$R\operatorname{Hom}_{\mathrm{D}_{\mathrm{qc}}(G_{\mathrm{pt}/\mathbb{A}^n})}(k, k) \cong k[u_1, \dots, u_n], \qquad \deg u_i = 2.$$

It is instructive to consider the $n = 1$ case. For the general computation we refer to [MI]. If $n = 1$, then $\Lambda \cong k \oplus k[1]$. Thus a free resolution of k is given by the complex

$$\cdots \to \Lambda[-2n] \to \cdots \to \Lambda[-4] \to \Lambda[-2] \to \Lambda \to k, \tag{3}$$

where the differentials map the element $1 \in \Lambda[2n]$ (sitting in degree $-2n$) to the element $\lambda \in \Lambda[2n-2]$ (sitting in degree $-2n+1$).

An application of the Barr–Beck–Lurie Theorem (Theorem 6.12, applied to the category of ind-coherent sheaves) gives the following description of the sheaves on $G_{\mathrm{pt}/\mathbb{A}^n}$.

Corollary 7.2. *There is a canonical equivalence of dg-categories*

$$\Psi\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(G_{\mathrm{pt}/\mathbb{A}^n}) \to \mathrm{D}^b_f(k[u_1, \dots, u_n]),$$

sending k to $k[u_1, \dots, u_n]$ and $\mathscr{O}_{G_{\mathrm{pt}/\mathbb{A}^n}} = \Lambda$ to k.

Note that since all u_i are in even degrees, we have forgetful functor from $k[u_1, \dots, u_n]$-modules to $k[x_1, \dots, x_n]$-modules (with $\deg x_i = 0$). Thus we define the support of a $k[u_1, \dots, u_n]$ as the support of the corresponding element of $\mathbf{QCoh}(\mathbb{A}^n)$. Since $\deg u_i = 2$, this support is always a conical subset of $\mathbb{A}^n$.

Since $\Psi(\Lambda) = k$ is supported on $0 \in \mathbb{A}^n$, the equivalence Ψ restricts to an equivalence

$$\mathbf{Perf}(G_{\mathrm{pt}/\mathbb{A}^n}) \to \mathrm{D}^b_f(k[u_1, \dots, u_n])_{\{0\}}, \tag{4}$$

where the latter category denotes the full subcategory of modules whose (set-theoretic) support is contained in $\{0\} \subset \mathbb{A}^n = \operatorname{Spec} k[u_1, \dots, x_n]$.

For any $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(G_{\mathrm{pt}/\mathbb{A}^n})$, we set $\operatorname{singsupp} \mathscr{F} = \operatorname{supp} \Psi(\mathscr{F}) \subseteq \mathbb{A}^n$, and call it the *singular support* of $\mathscr{F}$. We have thus obtained a non-trivial theory of support for sheaves on $G_{\mathrm{pt}/\mathbb{A}^n}$. By (4), we also achieved the goal of the introduction to this section: singular support measures how far $\mathscr{F}$ is from being perfect.

7.2. LOCAL COMPLETE INTERSECTIONS

Let us now study a more classical, but closely related, situation. We start with a local computation and let $X = \operatorname{Spec} R$ be a smooth affine scheme over a field k. Assume that $f_1, \dots, f_n$ is a regular sequence in R, so that $Z = \operatorname{Spec} R/(f_1, \dots, f_n)$ is of codimension n in X (or otherwise form the quotient as the dg scheme $\mathrm{pt} \times_{\mathbb{A}^n} X$). Set $A = \operatorname{Spec} R/(f_1, \dots, f_n)$.

We want to define some notion of singular support for coherent sheaves on Z. For this we take inspiration from the following construction by [BIK]: The graded center of a triangulated category $\mathbf{T}$ is

$$\operatorname{gr} Z_{\mathbf{T}} = \bigoplus_i \operatorname{Hom}(\mathrm{Id}_{\mathbf{T}}, \mathrm{Id}_{\mathbf{T}}[i])$$

By definition of a natural transformation, for any $A \in \mathbf{T}$ we obtain a natural map of rings

$$\operatorname{gr} Z_{\mathbf{T}} \to \operatorname{gr} \operatorname{End}(A) := \bigoplus_i \operatorname{Hom}_{\mathbf{T}}(A, A[i]),$$

turning $\operatorname{gr} \operatorname{End}(A)$ into a $\operatorname{gr} Z_{\mathbf{T}}$-algebra. Let now S be a commutative ring with a ring morphism $S \to \operatorname{gr} Z_{\mathbf{T}}$. Then $\operatorname{gr} \operatorname{End}(A)$ is naturally an S-module, and we can define its support as a subset of $\operatorname{Spec} S$. We will now construct such a ring S for $\mathbf{T} = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Z)$.

Remark 7.3. If $\mathbf{T}'$ is a dg-enhancement of $\mathbf{T}$, then the ring $\operatorname{gr} Z_{\mathbf{T}}$ is the cohomology of the Hochschild cohomology of $\mathbf{T}'$. Note also that the degree 0 part of $\operatorname{gr} Z_{\mathbf{T}}$ has a natural map to each $\operatorname{End}(A)$, and hence acts on each each object of $\mathbf{T}$.

Lemma 7.4. *For each $\mathscr{F} \in \mathbf{QCoh}(Z) = \mathbf{Mod}(A)$ there exist natural cohomological operators $\xi_1, \dots, \xi_n \in \operatorname{Ext}^2(\mathscr{F}, \mathscr{F})$. Thus $\operatorname{gr} \operatorname{End}(\mathscr{F})$ is a module over $A[\xi_1, \dots, \xi_n]$ with* $\deg \xi_i = 2$.

Proof. Let us construct ξ in the case of $n = 1$, i.e. $A = R/(f)$. Let $i\colon Z \hookrightarrow X$ be the inclusion and $\mathscr{F} \in \mathbf{QCoh}(Z)$. Resolving $\mathscr{O}_Z$ by $\mathscr{O}_X \xrightarrow{f} \mathscr{O}_X$ we see that

$$H^i(i^* i_* \mathscr{F}) = H^i(\mathscr{O}_Z \otimes_{\mathscr{O}_X} \mathscr{F}) \cong \begin{cases} \mathscr{F}, & i = 0, 1 \\ 0, & \text{otherwise.} \end{cases}$$

Hence there is a distinguished triangle[3]

$$\mathscr{F}[1] \to i^*i_*\mathscr{F} \to \mathscr{F}. \tag{5}$$

The corresponding element of $\mathrm{Ext}^2(\mathscr{F}, \mathscr{F})$ is the desired operator ξ. □

We note that since X is smooth, $i_*\mathscr{F}$ is perfect, and hence so is $i^*i_*\mathscr{F}$. Thus, iterating the triangle (5) one obtains a resolution of $\mathscr{F}$ by perfect complexes

$$(\cdots \to i^*i_*\mathscr{F}[4] \to i^*i_*\mathscr{F}[2] \to i^*i_*\mathscr{F}) \xrightarrow{\sim} \mathscr{F},$$

and hence

$$R\operatorname{Hom}(\mathscr{F}, \mathscr{F}) \cong R\operatorname{Hom}(\cdots \to i^*i_*\mathscr{F}[4] \to i^*i_*\mathscr{F}[2] \to i^*i_*\mathscr{F}, \mathscr{F}).$$

The construction above has an important consequence: If ξ is nilpotent on $\operatorname{gr}\operatorname{End}(\mathscr{F})$, then it factors through a finite truncation of the above complex, and hence $\mathscr{F}$ has a finite resolution by perfect complexes. So $\mathscr{F}$ is perfect. Conversely, if $\mathscr{F}$ is perfect, then $\mathrm{Ext}^i(\mathscr{F}, \mathscr{F}) = 0$ for $i \gg 0$, so ξ has to be nilpotent.

Theorem 7.5 ([G3, Theorem 3.1])**.** *If $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Z)$, then $\operatorname{gr}\operatorname{End}(\mathscr{F})$ is a finitely generated module over $A[\xi_1, \ldots, \xi_n]$.*

Definition 7.6. The *singular support* of $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Z)$ is

$$\operatorname{singsupp}\mathscr{F} := \operatorname{supp}_{A[\xi_1,\ldots,\xi_n]} \operatorname{gr}\operatorname{End}(\mathscr{F}) \subseteq \operatorname{Spec} A[\xi_1, \ldots, \xi_n] = Z \times \mathbb{A}^n.$$

Proposition 7.7. *For $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Z)$ we have the following properties of its singular support.*

(i) $\operatorname{supp}_A \operatorname{gr}\operatorname{End}(\mathscr{F}) = \operatorname{supp} F$*, i.e. the projection of* $\operatorname{singsupp}\mathscr{F}$ *onto Z is* $\operatorname{supp}\mathscr{F}$.

(ii) $\operatorname{singsupp}\mathscr{F}$ *is conical (because* $\operatorname{gr}\operatorname{End}(\mathscr{F})$ *is a graded module).*

(iii) $\mathscr{F}$ is perfect if and only $\operatorname{singsupp}\mathscr{F}$ *is (set-theoretically) contained in the zero section $Z \times \{0\} \subset Z \times \mathbb{A}^n$.*

[3]This triangle will in general not be trivial: in order to obtain the $\mathscr{O}_Z$-module structure on $i^*i_*\mathscr{F}$ we need to resolve $i_*\mathscr{F}$, rather than $\mathscr{O}_Z$. The resulting complex will have the same cohomology as the one obtained by resolving $\mathscr{O}_Z$, but may not be quasi-isomorphic as a complex of $\mathscr{O}_Z$-modules. For example, if $X = \mathbb{A}^1 = \operatorname{Spec} k[x]$, $Z = \operatorname{Spec} k[x]/(x^2)$, and $\mathscr{F} = k = k[x]/(x)$, then the two computations give the complexes

$$k[x]/(x^2) \otimes (k[x] \xrightarrow{x} k[x]) = k[x]/(x^2) \xrightarrow{x} k[x]/(x^2)$$

and

$$(k[x] \xrightarrow{x^2} k[x]) \otimes k[x]/(x) = k[x]/(x) \xrightarrow{0} k[x]/(x)$$

respectively. As complexes of k-vector spaces these are clearly quasi-isomorphic, but the morphism

$$\begin{tikzcd} k[x]/(x^2) \arrow[r, "x"] \arrow[d] & k[x]/(x^2) \arrow[d] \\ k[x]/(x) \arrow[r, "0"] & k[x]/(x) \end{tikzcd}$$

of complexes of $k[x]/(x^2)$-modules induces the zero morphism on H^0.

It turns out that singsupp $\mathscr{F}$ will always be contained in a specific subset of X, giving a sense to the use of “singular”. For this we consider the cotangent complex

$$\mathscr{O}_Z^n \xrightarrow{df_1,\dots,df_n} T^*X|_Z$$

of Z. Consider its (-1)st cohomology space “$H^{-1}(T^*Z)$”:

$$\begin{aligned}“H^{-1}(T^*Z)” &= \{(x, a_1, \dots, a_n) \in Z \times \mathbb{A}^n : \sum a_i df_i(x) = 0\} \\ &= \operatorname{Spec}(\operatorname{Sym} TZ[1]) \subseteq Z \times \mathbb{A}^n\end{aligned}$$

The scheme Z is smooth if and only if “$H^{-1}(T^*Z)$” $= Z$. Thus “$H^{-1}(T^*Z)$” is also called the *scheme of singularities* of Z.

Theorem 7.8 ([AG]). *The following statements hold:*

(i) For any $\mathscr{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Z)$ the singular support of $\mathscr{F}$ is contained in “$H^{-1}(T^*Z)$” $\subseteq X \times \mathbb{A}^n$.

(ii) These constructions can be made independent of the inclusion $i\colon Z \to X$.

(iii) The constructions are Zariski local, and hence can be defined for any local complete intersection scheme. (In fact, it is also smooth local.)

(iv) Singular support behaves well with respect to pushforward and pullback of sheaves.

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