Exotic sheaves and actions of quantum affine algebras

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Exotic sheaves

Consider $\mathbf{D}^b_{coh}(\tilde{\mathcal{N}})$ and $\mathbf{D}^b_{coh}(\tilde{\mathfrak{g}})$ and equivariant analogues. These categories are endowed with an action of the affine braid group of $\mathfrak{g}$ [BR].
Exotic sheaves are certain abelian subcategories of these categories that interact nicely with the braid group action ("braid positivity") and the pushforward to the base. They were most famously used by Bezrukavnikov and Mirković [BM] to prove Lusztig's conjectures on the canonical basis of the Grothendieck group of Springer fibers.
The category of exotic sheaves has nice properties, but is **hard to understand**: Both the existence of the braid group action and the exotic t-structure are highly non-obvious and require deep results from modular representation theory.

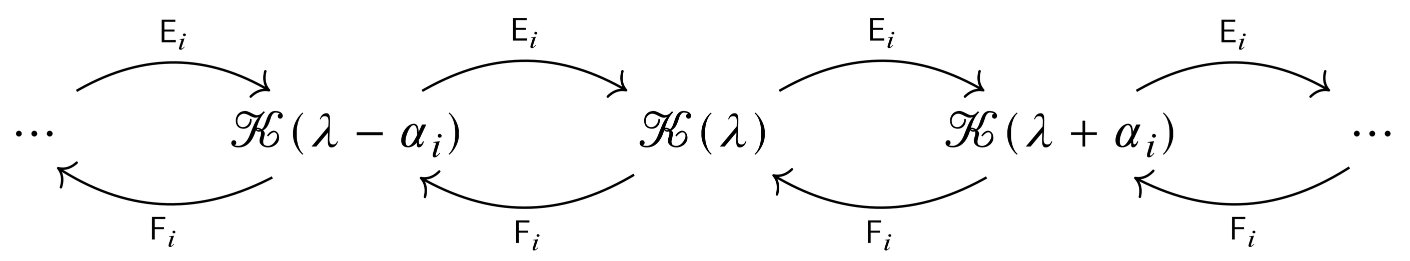
Our viewpoint

Exotic t-structures arise very naturally from categorical actions.

Categorical actions and braid groups

$\mathfrak{gl}_n$-action $\rightsquigarrow$ { weight spaces V_λ; action of e_i and f_i between them; relations (e.g. $[e_i,f_i]|_{V_\lambda} = \langle \alpha_i, \lambda\rangle \mathrm{Id}_{V_\lambda}$) }

categorical $\widehat{\mathfrak{gl}}_n$-action $\rightsquigarrow$ { $\lambda \mapsto$ triangulated category $\mathscr{K}(\lambda)$; bi-adjoint functors $\mathsf{E}_i, \mathsf{F}_i$ $(i = 0, \dots, n-1)$; categorified relations (e.g. $(*)$ below) }



$$\mathsf{E}_i\mathsf{F}_i|_{\mathscr{K}(\lambda)} = \mathsf{F}_i\mathsf{E}_i|_{\mathscr{K}(\lambda)} \oplus \bigoplus_{\langle\alpha_i,\lambda\rangle} \mathrm{Id}_{\mathscr{K}(\lambda)}, \qquad (*)$$

Typically: $\mathscr{K}(\lambda)$ are derived categories of sheaves and the functors are given by Fourier–Mukai kernels.
Out of the E_i, F_i one can naturally form complexes T_i giving an **action of the affine braid group** on $\bigoplus_\lambda \mathscr{K}(\lambda)$ [CK1].

Fineprint: Need some more data/constraints to make this work. The weight categories also have an important internal grading (hence the "quantum" in the title). The T_i are given by the Rickard complexes

$$\mathsf{T}_i|_{\mathscr{K}(\lambda)} = \mathsf{F}_i^{(\ell)} \to \mathsf{E}_i\mathsf{F}_i^{(\ell+1)} \to \mathsf{E}_i^{(2)}\mathsf{F}_i^{(\ell+2)} \to \cdots, \qquad \ell = \langle \alpha_i, \lambda\rangle.$$

Our philosophy

Weights: $\lambda = \underline{k} \in \mathbb{Z}^n$ such that $\sum k_i = n$. We restrict to level zero actions with all $k_i \geq 0$ (e.g. for $\widehat{\mathfrak{gl}}_2$ the roots are $\alpha_1 = (-1, 1)$ and $\alpha_0 = (1, -1)$).
In particular, we get an affine braid group action on the central category $\mathscr{K}(1, \dots, 1)$.

Main idea

categorical actions $\overset{\text{above}}{\rightsquigarrow}$ braid group actions

Maybe also:

categorical actions $\overset{?}{\rightsquigarrow}$ exotic sheaves

Typically it is easy to come up with interesting abelian subcategories at the highest weight $\mathscr{K}(n, 0, \dots, 0)$, e.g. one can use perverse-coherent sheaves. With the actions one should be able to get "matching" subcategories everywhere.

Theorem

If $\mathscr{K}(n, 0, \dots, 0)$ is "big enough":

abelian subcat. of $\mathscr{K}(n, 0, \dots, 0)$ $\overset{\mathsf{E}_i, \mathsf{F}_i \text{ restrict}}{\underset{\text{to exact functors}}{\rightsquigarrow}}$ abelian subcat. of each $\mathscr{K}(\underline{k})$

These subcategories are braid positive.

The main example

Define the varieties

$$\mathbb{Y}(\underline{k}) = \{\mathbb{C}[z]^n = L_0 \subseteq L_1 \subseteq \cdots L_n \subseteq \mathbb{C}(z)^n : zL_i \subseteq L_i,\ \dim(L_i/L_{i-1}) = k_i\},$$

and

$$\mathrm{Gr}^{\underline{\lambda}} = Y(\underline{k}) = \{\mathbb{C}[z]^n = L_0 \subseteq L_1 \subseteq \cdots L_n \subseteq \mathbb{C}(z)^n : zL_i \subseteq L_{i-1},\ \dim(L_i/L_{i-1}) = k_i\}.$$

These **convolution varieties** are well-studied and used, for example, to categorify link invariants or give a (quantum) K-theoretic analogue of the geometric Satake equivalence. Note that $Y(1, \dots, 1)$ has an open subvariety isomorphic to $\tilde{\mathcal{N}}$ and the $\mathbb{Y}(\underline{k})$ have open subvarieties isomorphic to partial **Grothendieck–Springer resolutions**.
The corresponding collections of derived categories $D^b(\mathbb{Y}(\underline{k}))$ and $D^b(Y(\underline{k}))$ each naturally carry categorical $\widehat{\mathfrak{gl}}_n$-actions.

Corollary

- Starting with perverse-coherent sheaves on $D^b(\mathbb{Y}(n, 0, \dots, 0))$ we get exotic sheaves on $D^b(\mathbb{Y}(1, \dots, 1))$.
- This restricts to exotic sheaves on $D^b(Y(1, \dots, 1))$.
- Restricting to the open subvarieties recovers the exotic sheaves of Bezrukavnikov–Mirković on $\tilde{\mathcal{N}}$, $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_{\mathscr{P}}$.

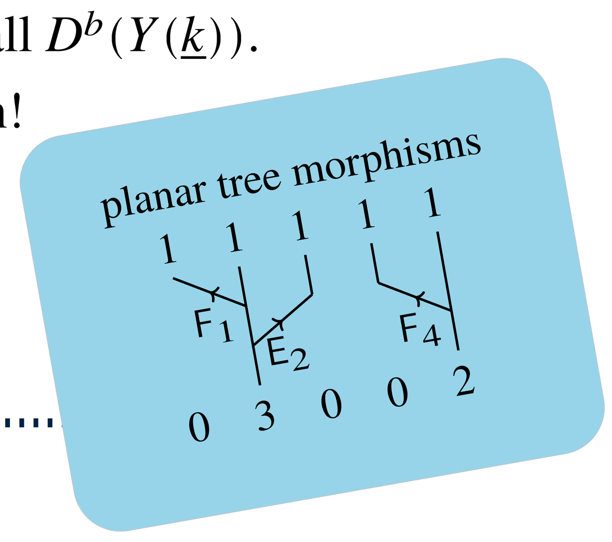
Applications

- No need for modular representation theory to obtain geometric results.
- Obtain categories of "exotic sheaves" on spaces where the known constructions (exceptional sets, tilting) do not apply.
- Of particular interest: exotic sheaves on convolution varieties of the affine Grassmannian (see example above).
- Can study exotic sheaves inductively, starting from simpler categories. For this we have the following converse theorem.

Theorem

braid pos. abelian subcat. of $\mathscr{K}(1, \dots, 1)$ $\overset{\text{planar trees restrict}}{\underset{\text{to exact functors}}{\rightsquigarrow}}$ abelian subcat. of each $\mathscr{K}(\underline{k})$

- In the example we get braid positive subcategories of all $D^b(Y(\underline{k}))$.
- $Y(n, 0, \cdots, 0) = \mathrm{pt}$ is a great starting point for induction!



Ongoing and future work

- Sheaves on more general convolution varieties.
- Structure results (weight structure, description of irreducibles, …).
- Applications to categorified knot invariants?
- Kac–Moody presentation (used in the theorem) versus loop presentation (more natural to define) of the $\widehat{\mathfrak{gl}}_n$-action.

References

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