Exotic sheaves and actions of quantum affine algebras

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Exotic sheaves

Consider the (equivariant) coherent derived categories of the (Grothendieck-) Springer resolution. These categories are endowed with an action of the affine braid group. The **exotic t-structures** are the unique t-structures on these categories such that

1. the positive braids act right t-exact, and
2. the pushforward to the base is t-exact.

They were most famously used by Bezrukavnikov and Mirković [BM] to prove (most of) Lusztig's conjectures on the canonical basis of the Grothendieck group of Springer fibers.

Exotic t-structures are hard to understand: Both the existence of the braid group action and the exotic t-structure are highly non-obvious and require deep results from modular representation theory.

Our viewpoint

Exotic t-structures arise very naturally from categorical actions.

Categorical actions and braid groups

$$\mathrm{GL}_n\text{-action} \rightsquigarrow \begin{cases} \text{weight spaces } V_\lambda \\ \text{action of } e_i \text{ and } f_i \text{ between them} \\ \text{relations (e.g. } [e_i,f_i]|_{V_\lambda} = \langle \alpha_i, \lambda\rangle \mathrm{Id}_{V_\lambda}) \end{cases}$$

$$\textbf{categorical } \widehat{\mathfrak{gl}}_n\textbf{-action} \rightsquigarrow \begin{cases} \lambda \mapsto \text{triangulated category } \mathscr{K}(\lambda) \\ \text{bi-adjoint functors } \mathsf{E}_i, \mathsf{F}_i \ (i = 0, \dots, n-1) \\ \text{categorified relations (e.g. } (*) \text{ below)} \end{cases}$$

$$\cdots \underset{\mathsf{F}}{\overset{\mathsf{E}}{\rightleftarrows}} \mathscr{K}(\lambda-2) \underset{\mathsf{F}}{\overset{\mathsf{E}}{\rightleftarrows}} \mathscr{K}(\lambda) \underset{\mathsf{F}}{\overset{\mathsf{E}}{\rightleftarrows}} \mathscr{K}(\lambda+2) \underset{\mathsf{F}}{\overset{\mathsf{E}}{\rightleftarrows}} \cdots$$

$$\mathsf{E}_i\mathsf{F}_i|_{\mathscr{K}(\lambda)} = \mathsf{F}_i\mathsf{E}_i|_{\mathscr{K}(\lambda)} \oplus \bigoplus_{\langle\alpha_i,\lambda\rangle} \mathrm{Id}_{\mathscr{K}(\lambda)}, \qquad (*)$$

Typically: $\mathscr{K}(\lambda)$ are derived categories of sheaves and the functors are given by Fourier–Mukai kernels.

Out of the E_i, F_i one can naturally form complexes T_i giving an action of the **affine braid group action** on $\bigoplus_\lambda \mathscr{K}(\lambda)$ [CK1].

Fineprint: Need some more data/constraints to make this work. The weight categories also have an important internal grading (hence the "quantum" in the title). The T_i are given by the Rickard complexes

$$\mathsf{T}_i|_{\mathscr{K}(\lambda)} = \mathsf{F}_i^{(\ell)} \to \mathsf{E}_i\mathsf{F}_i^{(\ell+1)} \to \mathsf{E}_i^{(2)}\mathsf{F}_i^{(\ell+2)} \to \cdots, \quad \ell = \langle \alpha_i, \lambda \rangle.$$

Results: inducing exotic t-structures

For our purposes, we identify the weights with n-tuples $\underline{k}$ of integers (e.g. for $\widehat{\mathfrak{sl}}_2$ the roots are $\alpha_1 = (-1,1)$ and $\alpha_0 = (1,-1)$). We assume that $\sum k_i = n$ and all $k_i \geq 0$. In particular, we get an affine braid group action on the central category $\mathscr{K}(1,\dots,1)$.

Main idea

$$\text{highest weight} \overset{\widehat{\mathfrak{gl}}_n\text{-action} +}{\underset{\mathbb{B}^{\mathrm{aff}}\text{-action}}{\rightsquigarrow}} \text{central weight}$$

Typically it is easy to come up with interesting t-structures on the highest weight $\mathscr{K}(n,0,\dots,0)$, e.g. one can use perverse-coherent t-structures.

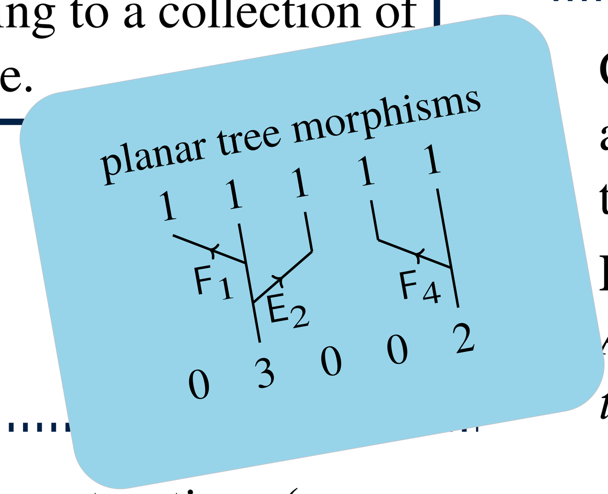
Theorem (Inducing from the highest weight)

Suppose that the highest weight category $\mathscr{K}(n,0,\dots,0)$ weakly generates $\mathscr{K}(1,\dots,1)$ under the $\widehat{\mathfrak{gl}}_n$-action. Then (under mild assumptions) there exits a unique extension to a t-structure on $\bigoplus_{\underline{k}} \mathscr{K}(\underline{k})$ such that all E_i and F_i are t-exact. Moreover, all positive braids act right t-exact with respect to this t-structure.

In other instances one has a t-structure on the central weight category and wants to spread it out to the other categories:

Theorem (Inducing from the central weight)

Suppose we are given a braid positive t-structure on $\mathscr{K}(1,\dots,1)$. Then there exists a unique t-structure on $\bigoplus_{\underline{k}} \mathscr{K}(\underline{k})$ determined by exactness of the 1-morphism $\psi : \mathscr{K}(\underline{k}) \to \mathscr{K}(1,\dots,1)$ corresponding to a collection of planar trees. Moreover, this t-structure is braid positive.



Applications and future work

- Obtain exotic t-structure on spaces where the known constructions (exceptional sets, tilting) do not apply.
- Of particular interest: exotic t-structures on convolution varieties of the affine Grassmannian (see example on the right). We will expand this to more general convolution varieties in future work.
- We expect that structural results (weight structure, description of irreducibles) can be obtained with our method and be applied to these new examples.

The main example

Define the varieties

$$\mathbb{Y}(\underline{k}) = \{\mathbb{C}[z]^n = L_0 \subset L_1 \subset \cdots L_n \subset \mathbb{C}(z)^n : zL_i \subseteq L_i,\ \dim(L_i/L_{i-1}) = k_i\},$$

and

$$\mathrm{Gr}^{\underline{\lambda}} = Y(\underline{k}) = \{\mathbb{C}[z]^n = L_0 \subset L_1 \subset \cdots L_n \subset \mathbb{C}(z)^n : zL_i \subseteq L_{i-1},\ \dim(L_i/L_{i-1}) = k_i\}.$$

These **convolution varieties** are well-studied and used, for example, to categorify link invariants or give a (quantum) K-theoretic analogue of the geometric Satake equivalence. Note that $Y(1,\dots,1)$ has an open subvariety isomorphic to $\tilde{\mathcal{N}}$ and the $\mathbb{Y}(\underline{k})$ have open subvarieties isomorphic to partial Grothendieck–Springer resolutions.

The corresponding collections of derived categories $D^b(\mathbb{Y}(\underline{k}))$ and $D^b(Y(\underline{k}))$ each naturally carry categorical $\widehat{\mathfrak{gl}}_n$-actions.

Corollary

- Starting with a perverse-coherent t-structure on $D^b(\mathbb{Y}(n,0,\dots,0))$ we get an exotic t-structure on $D^b(\mathbb{Y}(1,\dots,1))$.
- This restricts to a perverse t-structure on $D^b(Y(1,\dots,1))$.
- This induces a braid positive t-structure on all $D^b(Y(\underline{k}))$.
- Restricting to the open subvarieties recovers the exotic t-structures of Bezrukavnikov–Mirković on $\tilde{\mathcal{N}}$, $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_{\mathscr{P}}$.

How?

Careful analysis of the structure and combinatorics of categorical $\widehat{\mathfrak{gl}}_n$-actions and the associated braid group actions allows us to induce t-structures using the following lemma (which is based on a theorem of Polishchuk [P]).

Lemma. *Let $\Phi : D^b(X) \to D^b(Y)$ be a conservative Fourier–Mukai functor. Assume that we are given a t-structure on $D^b(Y)$ such that $\Phi \circ \Phi^L$ is right t-exact. Then there exists a unique t-structure on $D^b(X)$ such that Φ is t-exact.*

References

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